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On trade-off solution pairs in a special type of transportation problem

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Abstract. In this paper, we have developed an algorithm to determine the trade-off solution pairs in a special type of transportation problem considering two objectives, *viz.*, cost and time. It is assumed that the time of transportation from an origin to a destination depends on the quantity transported. This results in a time objective, which is a piecewise constant increasing function. The cost objective function is taken to be linear. A potential physical situation of this model is given and a numerical example is worked out to illustrate the algorithm.

Keywords. Trade-off solution pairs; transportation problem; time objective.

1. Introduction

The time minimizing transportation problem (TMTP), also known as Bottleneck Transportation Problem, was first studied by Hammer [11, 12]. Garfinkel and Rao [9] proposed a threshold algorithm to solve TMTP. Szware [19] refined Hammer's algorithm and also suggested a revision of the proof of a result in Hammer's paper [11]. He also provided a thorough survey of the earlier East European literature on TMTP. Bhatia *et al* [3], Derigs and Zimmermann [6], Finke and Smith [7], Sharma and Swarup [16] provide various algorithms for solving TMTP. Seshan and Tikekar [15] pointed out an error in a step in the algorithm given by Sharma and Swarup [16]. Bhatia [5] has developed a method of solving zero-one time minimization transportation problem. A computational study of this problem was done by Foulds and Gibbons [8]. Isermann [13] presented an algorithm to enumerate all efficient solutions for a linear multiple objective transportation problem.

The fundamental assumption in a TMTP is that the time for transporting a certain amount of goods from a source to a destination is independent of the amount of goods transported. Achary and Seshan [1] replace the assumption by a more general and realistic assumption that the time of transportation over a route depends on the amount of shipment through that route. This assumption leads to a piecewise constant increasing time objective function.

When one considers a transportation problem with two objectives, say, cost and time, one fails to get an optimum solution satisfying both the objectives. In such cases, we see for a sequence of solutions termed as trade-off (efficient, Pareto-optimal, non-dominated) solutions. Methods of obtaining trade-off solutions have been developed by Glickman and Berger [10], Bhatia *et al* [4] and Srinivasan and Thompson [18]. Aneja and Nair [2] developed an algorithm to solve a Bicriteria Transportation Problem having both objective functions linear. Sharma and Swarup [17] provide an

algorithm for obtaining trade-off solutions in a multi-dimensional transportation problem.

In this paper, we develop an algorithm for obtaining trade-off solutions in a special type of transportation problem. We consider the usual linear cost objective along with the time objective considered in Achary and Seshan [1]. Section 2 of this paper deals with the mathematical formulation of the problem and provides a potential physical situation in support of the mathematical model. In § 3, some definitions and the methodology of the algorithm are given followed by an algorithm. A numerical example, illustrating the algorithm, is worked out in § 4.

2. The problem

Consider a transportation problem with m sources and n destinations. Let a_i ($i = 1, 2, \ldots, m$) be the availability at source i, b_j ($j = 1, 2, \ldots, n$), the demand at destination j and U_{ij}, the capacity restriction over the route (i, j). The constraints of the problem then are

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ for all } i. \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ for all } j. \tag{2}$$

$$0 \leqslant x_{ij} \leqslant U_{ij} \text{ for all } i, j. \tag{3}$$

The cost objective function is given by

$$\min C(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}. \tag{4}$$

Where c_{ij} is the unit cost of transportation over the route (i, j). We consider the time objective function

$$\min T(X) = \max \{t_{ij}(x_{ij})\} \tag{5}$$

where $t_{ij}(x_{ij})$ is defined in (6) below.

For each route (i, j) let the interval $[0, U_{ij}]$ be divided into subintervals as follows.

$$0 = \mu_{ij}^0 < \mu_{ij}^1 < \ldots < \mu_{ij}^{r(i,j)} = U_{ij}$$

and let $t_{ij}^1 < t_{ij}^2 < \ldots t_{ij}^{r(i,j)}$.

Then

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if} \quad x_{ij} = 0 \\ t_{ij}^k & \text{if} \quad \mu_{ij}^{k-1} < x_{ij} \leqslant \mu_{ij}^k \\ & (1 \leqslant k \leqslant r(i,j)) \end{cases} \tag{6}$$

It is not necessary that all the intervals $[0, U_{ij}]$ are divided into the same number of subintervals. Without loss of generality we can assume that $U_{ij} \leqslant \min(a_i, b_j)$ for all (i, j). We shall call the problem defined by (1) through (5) as problem P.

Such a problem can arise in a practical situation described below. Let a single mode of transportation be considered for the shipment of goods from any source i to a destination j with the unit cost of transportation c_{ij}. Suppose that there is a capacity limitation m_{ij} for the amount that can be transported from the ith source to the jth destination in a fixed time interval T_{ij}. Let t_{ij} be the time for transporting up to and including m_{ij} units from the ith source to the jth destination. Then $t_{ij}(x_{ij})$ has the form

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if} \quad x_{ij} = 0 \\ kT_{ij} + t_{ij} & \text{if} \quad km_{ij} < x_{ij} \leqslant (k+1)m_{ij}, k \geqslant 0 \end{cases} \tag{7}$$

If $x_{ij} = km_{ij} + y_{ij}$, $0 \leqslant y_{ij} < m_{ij}$, then transportation of m_{ij} units starts at each of the time points $0, T_{ij}, \ldots, (k-1)T_{ij}$ and the transportation of remaining y_{ij} units starts at time kT_{ij}. The cost objective is linear in this case. The function $t_{ij}(x_{ij})$ defined by (6) is a generalisation of $t_{ij}(x_{ij})$ considered in this example.

3. Definitions and methodology

3.1. *Definitions*

3.1a. *Definition* 1: T is said to be a feasible time for the problem P, if there exists a feasible solution X for the problem P, with $T(X) \leqslant T$. Otherwise T is said to be an infeasible time for the problem P.

3.1b. *Definition* 2: Let X be a feasible solution for P which is also cost optimal. Let $T(X) = T$ be the feasible time, and $C(X) = C$, the optimal cost. The pair (C, T) is called a solution pair.

3.1c. *Definition* 3: A solution pair (C, T) is said to be an efficient solution pair (trade-off solution pair, Pareto Optimal solution pair, nondominated solution pair) if there exists no other solution pair (C', T') such that (i) $C' < C$ and $T' \leqslant T$ or (ii) $C' \leqslant C$ and $T' < T$.

Given a time T, its feasibility can be checked as follows:

Define

$$c'_{ij} = \begin{cases} c_{ij} & \text{if} \quad t^k_{ij} \leqslant T \quad \text{for} \quad k \geqslant 1 \\ M \text{ if } t^1_{ij} > T \end{cases} \tag{8}$$

$$U'_{ij} = \begin{cases} \mu^k_{ij} & \text{if} \quad t^k_{ij} \leqslant T < t^{k+1}_{ij}, k \geqslant 1 \\ U_{ij} & \text{if} \quad t^1_{ij} > T. \end{cases} \tag{9}$$

Consider the following cost minimising transportation problem (CMTP):

Minimize $Z(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij} x^{ij}$

subject to (1), (2) and

$$0 \leqslant x_{ij} \leqslant U'_{ij} \tag{10}$$

If the problem has a feasible solution with a finite optimal value Z, then the time T is feasible time. Otherwise it is an infeasible time.

3.2. *Methodology of the algorithm*

The algorithm starts with a cost optimal solution. $X_1 = (x_{ij})$ for the transportation problem defined by (1) through (4). The feasible time T_1 corresponding to this solution is obtained using (5) and solution pair (C_1, T_1) is recorded. Next, we determine a time $T' = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_1\}$ and check for its feasibility, using the technique discussed in 3.1. The details of solving the bounded transportation problem are given in Murty [14]. The feasible solution obtained with a finite optimal cost may have a feasible time smaller than T'. In that case, the better value is considered and the solution pair is recorded. This step is repeated until the algorithm establishes the infeasibility of a given time. At each iteration, the optimal solution of the previous iteration is taken as a starting solution for arriving at a feasible solution in the current iteration. The sequence of solution pairs obtained is examined for redundant solutions *i.e.* those solution which have the same optimal cost but different feasible times. Among such solution pairs, only those solution pairs having a smaller (better) feasible time are retained and the final sequence of trade-off solution pairs is recorded.

The algorithm terminates in a finite number of steps, since only a finite number of distinct T values are to be checked for their feasibility.

3.3. *The Algorithm*

STEP 1:

Set $r = 1$. Consider the cost minimising transportation problem in P. Let $X_1 = (x_{ij})$ be an optimal solution and $C(X_1) = C_1$, the optimal value. Find $T_1 = \max \{t_{ij}(x_{ij}) | x_{ij} > 0\}$. Record (C_1, T_1) and go to step 2.

STEP 2:

Set r to $r+1$. Find $T_r' = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_{r-1}\}$. Define c_{ij}' and U_{ij}' as in (8) and (9). Solve CMTP given in (10). If it has a finite optimal value go to step 3. Otherwise go to step 4.

STEP 3:

Let $X_r = (x_{ij})$ be an optimal solution with $C(X_r) = C_r$ and $T(X_r) = T_r$. Record (C_r, T_r) and go to step 2.

STEP 4:

The time T_r' is an infeasible time. Search the sequence of solution pairs for redundant solutions, if any, and eliminate them from the sequence.

4. Numerical example

Consider a numerical example with the data given in tables 1 and 2. The last column in table 1 gives the availabilities $(a_i, i = 1, 2, 3, 4)$ and the last row gives the demands $(b_j, j = 1, 2, \ldots 5)$. Every (i, j)th cell contains t_{ij}^k's and the corresponding intervals $(\mu_{ij}^{k-1}, \mu_{ij}^k)$. Table 2 gives the cost matrix (c_{ij}). In tables 3–5, the solutions corresponding to the trade-off solution pairs are shown. The upper bounds of the variables are shown,

Table 1. Basic data for the example.

					a_i
8(0–25) 10(25–40) 12(40–55)	6(0–30) 9(30–60) 12(60–80)	8(0–15) 10(15–30)	9(0–20) 11(20–35)	11(0–25) 13(25–35) 15(35–50)	90
6(0–20) 9(20–35)	5(0–25) 8(25–35)	6(0–15) 7(15–30)	6(0–20) 8(20–35)	7(0–20) 9(20–35)	35
8(0–25) 11(25–40) 13(40–55)	7(0–20) 9(20–40) 10(40–60)	8(0–15) 10(15–30)	4(0–20) 6(20–35)	15(0–30) 17(30–50)	60
6(0–30) 9(30–40) 11(40–55)	5(0–25) 7(25–45) 9(45–65)	5(0–15) 7(15–30)	13(0–20) 15(20–35)	11(0–30) 13(30–50)	65
b_j 55	80	30	35	50	

Table 2. Cost data for the example.

4	3	7	8	6
2	10	11	5	9
7	10	8	5	6
3	8	1	11	2

Table 3. Iteration 1.

(10)	(80)			
(35)				
			(35)	(25)
(10)		(30)		(25)

Table 4. Iteration 2.

	(55)			35 (35)
(20)				(15)
	(25)		(35)	M
(35)		(30)	20	30

Table 5. Iteration 3.

40 (5)	60 60			25 25
(10)				(25)
40 (5)	(20)		(35)	M
(35)		(30)	M	30

wherever necessary, at the right top corner of the corresponding cells. The values of th basic variables at a solution are bracketed.

ITERATION 1:

The cost minimizing problem is solved. The cost optimal solution X_1 is shown in table 3

$$C(X_1) = C_1 = 785 \quad T(X_1) = T_1 = \max\{t_{ij}(x_{ij}) | x_{ij} > 0\} = 15.$$

$$(C_1, T_1) = (785, 15).$$

ITERATION 2:

$T'_2 = \max\{t_{ij}^k | t_{ij}^k < 15\} = 13$. The modified CMTP is solved. The cost optimal feasibl solution is shown in table 4. $C(X_2) = 1110, T(X_2) = 13$ and the efficient pair is (1110 13).

ITERATION 3:

$T'_3 = \max\{t_{ij}^k | t_{ij}^k < 13\} = 11$. The optimal solution of the modified CMTP is shown i table 5. $(C_3, T_3) = (1505, 11)$.

ITERATION 4:

$T'_4 = \max\{t_{ij}^k | t_{ij}^k < 11\} = 10$. At this iteration we find that 10 is an infeasible time There are no redundant solutions. The sequence of efficient solution pairs is (785, 15 (1110, 13) (1505, 11).

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Velocity and temperature distribution in a system consisting of a fluid layer overlying a layer of porous medium

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Abstract. The velocity and temperature distribution in a system consisting of a fluid layer overlying a layer of porous medium is investigated in the presence of buoyancy and surface tension forces. The analysis indicates that buoyancy and surface tension forces are additive and aid the flow by counteracting the effect of permeability. The temperature distribution is sensitive to the variation of the aspect ratio, and conductivities of the media. Further, the inclusion of the viscous dissipation term markedly affects the temperature distribution.

Keywords. Permeable; Navier-Stokes equation; Brinkman model; Boussinesq approximation; slip parameter; surface tension; coupled flow.

1. Introduction

Convection, forced and/or free, in a porous medium saturated with the same fluid is of interest in geophysics, oil recovery techniques, cryogenic industry, space applications, cooling problems and thermal insulation in high temperature reactor technology. The study of flow past and/or through the porous medium has therefore been attracting increasing attention. In recent years there has been a copious literature dealing with the different aspects of the above problem.

Nield [8] investigated the problem of onset of convection in a fluid layer overlying a layer of porous medium and discussed the effect of surface tension in inhibiting the convective instability. Chandrasekhara *et al* [4] investigated the convective instability in a fluid layer overlying a layer of porous medium using Brinkman momentum equation. They have shown that the viscous resistance term is dominant at low values of σ, the porous parameter, and thereby increases the critical Rayleigh-Darcy number. The present investigation is devoted to the study of simple one-dimensional coupled flows in a system consisting of a fluid layer overlying a layer of porous medium to understand the effect of buoyancy and the surface tension forces on the coupled flows when the temperature varies linearly in the direction of flow.

The governing equations of momentum and energy are solved separately for the fluid layer and for the porous medium and are matched at the nominal surface using the continuity conditions of velocity, temperature and heat flux. The solution for the porous medium is obtained by direct and singular perturbation methods. The analysis indicates that the effects of surface tension and buoyancy are additive. It is also observed that the viscous dissipation term is very dominant in affecting the temperature profiles.

2. Formulation of the problem and boundary conditions

The physical model (figure 1) consists of a fluid layer occupying the region $0 \leqslant$ $\leqslant h_1$, hereafter referred to as zone-I and a porous medium saturated with the same flui occupying the region $-h_2 \leqslant y \leqslant 0$ hereafter referred to as zone-II. The fluid in zone and zone-II is driven by a constant pressure gradient applied in the streamwis direction. Buoyancy and surface tension forces are caused by the entry temperatu having a constant temperature gradient in the flow direction. To analyse the proble we assume that Oberbeck-Boussinesq equations are valid in the fluid layer an Brinkman-Oberbeck-Boussinesq equations are valid in the porous medium. Th necessity of using Brinkman-Oberbeck-Boussinesq equation in the porous mediu instead of Darcy-Oberbeck-Boussinesq equation arises due to the fact that the Darc equation is valid in regions far away from the boundaries and hence it is applied to larg systems where the boundary layer flow does not affect the main body of the fluid. But i a system of finite width the existence of a boundary layer beneath the nominal surfac (Beavers and Joseph [1]) due to the presence of slip velocity and the presence of a rigi wall dictate the necessity of boundary layer type of equation of the Brinkman for (Brinkman [2]; Tam [10]; Lundgren [6]; Prabhamani and Rudraiah [9 Chandrasekhara and Vortmeyer [3]).

To derive the basic equations, we make the following assumptions. (i) The flow steady and fully developed so that all the physical quantities except pressure an temperature are functions of y only. (ii) The porous medium is homogeneous an isotropic so that its permeability K is constant. (iii) The thickness of the fluid layer the same as that of the porous layer (*i.e.*, $h_1 \doteq h_2 = h$). The basic equations valid zone-I and zone-II are:

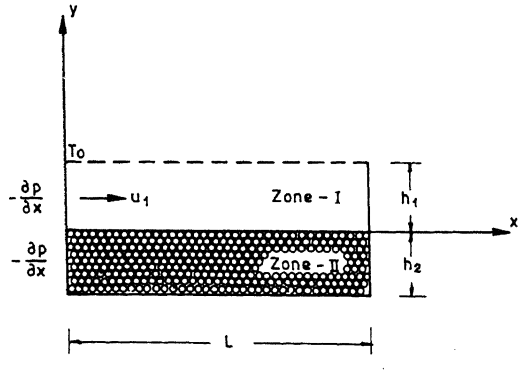
ZONE-I

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}, \tag{}$$

$$\partial p/\partial y = -\rho g, \tag{}$$

$$\rho = \rho_0[1-\beta(T_1'-T_A)], \tag{}$$

$$\rho_0 c_p u_1 \frac{\partial T_1'}{\partial x} = k_1 \frac{\partial^2 T_1'}{\partial y^2} + \mu(\partial u_1/\partial y)^2. \tag{}$$



ZONE-II

$$\partial^2 u_2/\partial y^2 - u_2/K = 1/\mu\, \partial p/\partial x, \tag{5}$$

$$\partial p/\partial y = -\rho g, \tag{6}$$

$$\rho_0 c_p u_2 \frac{\partial T_2'}{\partial x} = k_2 \frac{\partial^2 T_2'}{\partial y^2} + \mu\left(\frac{\partial u_2}{\partial y}\right)^2, \tag{7}$$

where u_1, u_2 = velocity in the axial direction; p = pressure; ρ = density; T_1', T_2' = temperature; μ = viscosity; c_p = specific heat at constant pressure; K = permeability; k_1 = thermal conductivity of the fluid; k_2 = average conductivity of the porous medium and is defined by $\varepsilon k_1 + (1-\varepsilon)k_s$; k_s = conductivity of the solid; ε = porosity; ρ_0 = density at $T' = T_A$; β = thermal expansion coefficient; T_A = ambient temperature; g = acceleration due to gravity.

Eliminating the pressure p in (1) using (2) and (3) we get

$$\frac{\partial^3 u_1}{\partial y^3} = \frac{g\beta}{\nu}\frac{\partial T_1'}{\partial x}. \tag{8}$$

Equation (8) describes the fully developed flow only when the right side is independent of x *i.e.*, heat flux in the direction of flow must be constant. To satisfy this condition the temperature T' should be of the form

$$T_1'(xy) = Ax + T_1(y), \tag{9}$$

where $T_1(y)$ is the entrance temperature and A is the axial temperature gradient. Further, we impose the condition that

$$T_1'(0, h) = T_1(h) = T_0. \tag{9a}$$

Similarly the momentum equation in zone-II takes the form

$$\frac{\partial^3 u_2}{\partial y^3} - \frac{1}{K}\frac{\partial u_2}{\partial y} = \frac{g\beta}{\nu}\frac{\partial T_2'}{\partial x}. \tag{10}$$

Introducing the following non-dimensional variables

$$u^* = u/Q, \quad T_1^* = T_1' - T_A/T_0 - T_A, \quad \sigma = h/\sqrt{K},$$

$$\eta = y/h, \quad T_2^* = T_2' - T_A/T_0 - T_A,$$

$$\bar{Q} = -\frac{h^2}{\mu}\frac{\partial p}{\partial x}, \quad x^* = x/L,$$

and removing the asterisks for convenience, the momentum and energy equation in zone-I and zone-II are written in the form:

ZONE-I

$$\partial^3 u_1/\partial\eta^3 = -N_0, \tag{11}$$

$$(\partial^2 T_1/\partial\eta^2) + p_1 \cdot E(\partial u_1/\partial\eta)^2 = P_{e1}\, aA' u_1. \tag{12}$$

ZONE-II

$$\frac{\partial^3 u_2}{\partial\eta^3} - \sigma^2\frac{\partial u_2}{\partial\eta} = -N_0, \tag{13}$$

$$(\partial^2 T_2/\partial\eta^2) + P_2 \cdot E(\partial u_2/\partial\eta)^2 = P_{e2} a A' u_2, \tag{1}$$

where $\Delta T = T_0 - T_A$, T_A = ambient temperature; A' = nondimensional temperatu gradient, $\partial T/\partial x$; P_{e1} and P_{e2} = Pecklet numbers, $\bar{Q}h/k_1^*$ and $\bar{Q}h/k_2^*$; k_1^* and k = thermal diffusivity, $(k_1/\rho_0 c_p,\ k_2/\rho_0 c_p)$; P_1 and P_2 = Prandtl numbe $(\nu\rho c_p/k_1, \nu\rho c_p/k_2)$; E = Eckert number, $(\bar{Q}^2/\Delta T \cdot c_p)$; $N_0 = (g\beta h^3/\bar{Q}\nu)(\Delta T/L$ $(-\partial T/\partial x)$.

The values of P_{e1}, P_{e2}, P_1 and P_2 depend on the k_1 and k_2 values.

In the above nondimensionalised equations we have chosen $\bar{Q} = -(h^2/\mu)(\partial p/\partial x$ $= Q\sigma^2$ as the reference velocity, which is the product of Darcy velocity and the squa of the porous parameter σ, since both zones are subjected to same horizontal pressu gradient. The length of the porous bed L is taken as the reference length in the X direction.

3. Boundary conditions

To solve the above equations proper boundary conditions on velocity and temperatu have to be specified. The velocity boundary conditions are:

$$\partial u_1/\partial\eta = M_0 \quad \text{at} \quad \eta = 1, \tag{1}$$

$$\partial^2 u_1/\partial\eta^2 = -1 \quad \text{at} \quad \eta = 0, \tag{1}$$

$$\partial u_1/\partial\eta = \alpha\sigma[u_B - (1/\sigma^2)] \quad \text{at} \quad \eta = 0, \tag{1}$$

$$u_2 = u_B \quad \text{at} \quad \eta = 0, \tag{1}$$

$$(\partial^2 u_2/\partial\eta^2) - \sigma u_2 = -1 \quad \text{at} \quad \eta = 0, \tag{1}$$

$$u_2 = 0 \quad \text{at} \quad \eta = -1, \tag{2}$$

where

$$M_0 = M/P_{e1}(-\partial T_1/\partial x); \text{ modified Marangoni number,}$$

$$M = -\frac{\sigma_T}{\mu}\frac{h \cdot \Delta T}{k}; \text{ Marangoni number,}$$

$$\Delta T = (T_0 - T_A).$$

Boundary condition (15) represents the continuity of tangential stress at the upper fre surface. Equations 1 and (19) express the fact that the flow in zone-I and zone-II is drive by the same pressure gradient. Equations (17) and (18) represent the slip boundar conditions of Beavers and Jospeh [1]. Equation (20) is the no-slip condition.

The temperature boundary conditions are:

$$T_1 = 1 \quad \text{at} \quad \eta = 1, \tag{2}$$

$$\partial T_1/\partial\eta = B(\partial T_2/\partial\eta) \quad \text{at} \quad \eta = 0, \tag{2}$$

$$T_1 = T_2 = T_B \quad \text{at} \quad \eta = 0, \tag{2}$$

$$T_2 = 0 \quad \text{at} \quad \eta = -1, \tag{2}$$

where

$$B = k_2/k_1.$$

Equation (21) is obtained from (9) and it specifies the entry temperature at $\eta = 1$ and $x = 0$. Equation (22) expresses the continuity of heat flux at the nominal surface; and (23) expresses the continuity of temperature at the nominal surface.

3.1. Velocity distribution

In this section analytical expressions for velocity distribution in zone-I and zone-II are obtained and its behaviour is discussed.

The expression for the velocity distribution for zone-I using (11), (15), (16) and (17) is obtained in the form

$$u_1 = -N_0\frac{\eta^3}{6} - \frac{\eta^2}{2} + \left(M_0 + \frac{N_0}{2} + 1\right)\eta + u_B \tag{25}$$

and

$$u_B = \frac{M_0 + (N_0/2) + 1}{\alpha\sigma} + \frac{1}{\sigma^2} \tag{26}$$

where u_B is the slip velocity at the nominal surface.

To understand the influence of slip velocity, buoyancy and surface tension on flow characteristics it is useful to calculate the mass flow rate in the fluid layer. If M_p denotes the nondimensional mass flow rate per unit width, then

$$M_p = \int_0^1 u_1 \, d\eta = \frac{5N_0}{24} + \frac{1}{3} + \frac{M_0}{2} + u_B. \tag{27}$$

On the other hand, if the fluid layer is bounded by impermeable walls the mass flow rate M_I is given by

$$M_I = (5N_0 + 8)/24. \tag{28}$$

It may be noted that the net flow M_I in impermeable walls can be stopped by setting $5N_0 = -8$. The ratio of M_p/M_I can be calculated for the conditions of equal pressure gradient and equal thickness of layers from (28) and (29) and has the form

$$\frac{M_p}{M_I} = 1 + \frac{12M_0}{5N_0 + 8} + \frac{[\sigma(M_0 + (N_0/2) + 1) + \alpha]24}{\alpha\sigma^2(5N_0 + 8)} \tag{29}$$

M_p/M_I ratios are evaluated for different values of σ, M_0 and N_0 and is represented in figure 2. It is observed that for low values of σ increase in M_0 the mass flow rate increases by large amounts. For example, $M_0 = N_0 = 1$, $\sigma = 2$ and $\alpha = 0{\cdot}1$, $M_p/M_I = 25{\cdot}4645$, and $M_0 = 5$, $N_0 = 1$, $\sigma = 2$, $\alpha = 0{\cdot}1$, $M_p/M_I = 66{\cdot}0849$. From the above results we can see that surface tension effect is very dominant in increasing the mass flow rate. Hence to obtain greater mass flow rate M_0 should be large. As $\sigma \to \infty$ the ratio M_p/M_I asymptotically approaches the value of,

$$\frac{M_p}{M_I} = 1 + \frac{12M_0}{5N_0 + 8}.$$

The velocity distribution for zone-II is obtained both by direct and singular perturbation methods. The solution obtained by direct method is of the form,

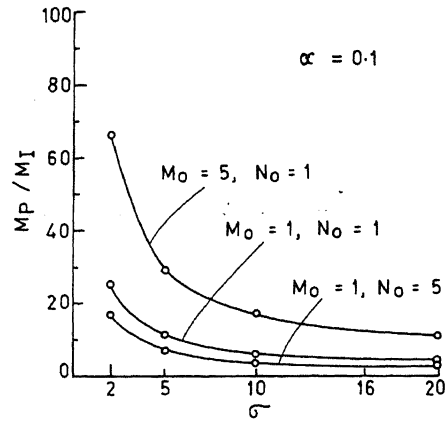


Figure 2. Mass flow rate *vs* porous parameter.

$$u_2 = \frac{1}{\sigma^2}\left[(u_B\sigma^2 - 1)\cosh\sigma\eta + \frac{\sinh\sigma\eta}{\sinh\sigma}\{(u_B\sigma^2 - 1)\cosh\sigma - N_0 + 1\} + N_0\eta + 1\right]. \tag{30}$$

The above solution is valid for arbitrary values of σ. The boundary layer type of solution valid for large values of σ is obtained by singular perturbation method and it is discussed below.

Dividing (13) by σ^2 and writing $1/\sigma^2$ equal to ε^2 we obtain

$$\varepsilon^2\frac{\partial^3 u_2}{\partial\eta^3} - \frac{\partial u_2}{\partial\eta} = -N_0\varepsilon^2. \tag{31}$$

Assuming that, in the limit $\varepsilon \to 0$; $N_0\varepsilon^2$ remains finite, we find that the highest derivative of (31) is lost and the order of the equation reduces from third order to first order. This reduced order equation cannot satisfy all the boundary conditions and hence its solution is not uniformly valid in the entire region of interest. Therefore, we develop inner and outer solutions valid in their respective domins of validity and match them in the overlap domain.

3.2. *Outer solution*

In the limit σ tends to zero (31) reduces to

$$\partial u_2/\partial\eta = N, \tag{32}$$

where $N = N_0\varepsilon^2$.

The solution of (32) is given by

$$u_2 = N\eta + c. \tag{33}$$

Imposing the outer boundary conditions $u_2 = 0$ at $\eta = -1$ we obtain the outer solution in the form

$$u_2(0) = N(\eta + 1). \tag{34}$$

Solution (34) does not satisfy all the boundary conditions and hence is not uniformly valid in the entire flow region. In order to get a uniformly valid solution we develop the inner solution by introducing the stretching variable $\eta^* = \eta/\varepsilon$. Expressing (13) in terms of η^* we have

$$\frac{\partial^3 u_2(i)}{\partial \eta^{*3}} - \frac{\partial u_2(i)}{\partial \eta^*} = -N\varepsilon. \tag{35}$$

Equation (35) in the limit $\varepsilon \to 0$ reduces to the form

$$\frac{\partial^3 u_2(i)}{\partial \eta^{*3}} - \frac{\partial u_2(i)}{\partial \eta^*} = 0. \tag{36}$$

The solution of (36) satisfying the boundary conditions $u_2 = u_B$ at $\eta^* = 0$ is

$$u_2(i) = u_B - D + De^{\eta^*}.$$

In the above solution, the constant D is still undetermined and is obtained by matching the inner and outer solutions in the overlapping domain. The matching condition after Cole (5) and Lagerstrom and Casten [7] is

$$u_2(0) = u_2(i)(-\infty). \tag{37}$$

The inner solution after matching is

$$u_2(i) = N + (u_B - N)e^{\eta^*}. \tag{38}$$

The uniform solution valid in the region of interest is then given by

$$u_2 = \frac{N_0}{\sigma^2}(\eta + 1) + \left(u_B - \frac{N_0}{\sigma^2}\right)e^{\eta\sigma}. \tag{39}$$

Equation (39) satisfies both the boundary conditions at 0 and -1 except for a transcendentally small term $(u_B - N_0/\sigma^2)e^{-\sigma}$. From (39) it is seen that the thickness of the boundary layer is of the order $\eta = 1/\sigma$ and is not provided by the solution (30) obtained by the direct method. The two solutions (30) and (39) are in excellent agreement for $\sigma = 5$.

To illustrate the effects of buoyancy, surface tension and porous parameter σ, on the velocity distribution, expressions (25), (30) and (39) are numerically evaluated for different values of α, σ, N_0 and M_0 and the results are represented in figures 3 and 4. It is observed that for low values of σ, when M_0 and N_0 are kept constant, the magnitude of velocity is large. However, as σ is increased the velocity is greatly reduced in magnitude due to Darcy resistance term and the shape of the profiles appears to be same. Hence, the permeability is the dominant factor in controlling the flow. For a given value of σ increase in M_0 and N_0 the velocity increases. This shows that both buoyancy and surface tension forces are additive and aid the flow. These two forces thereby counteract the effect of permeability. When M_0 and N_0 are negative there is a back flow in both the zones which is not represented in the figure.

3.3. *Temperature distribution*

The expressions for the temperature distribution in zone-I and zone-II are obtained by solving (12) and (14) using the boundary conditions (21)–(24), in the following form:

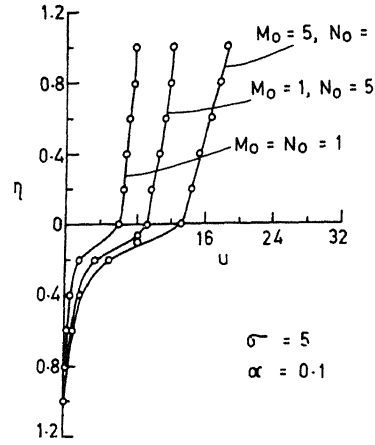


Figure 3. Velocity distribution *vs* η.

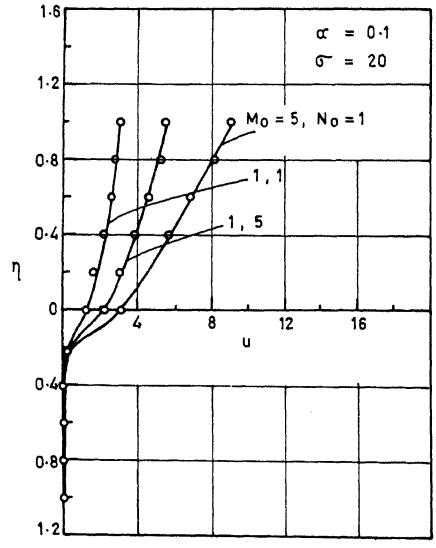


Figure 4. Velocity distribution *vs* η.

ZONE-I

$$T_1 = P_{e1} aA' \left(-\frac{N_0 \eta^5}{120} - \frac{\eta^4}{24} + \frac{b_1 \eta^3}{6} + \frac{u_B \eta^2}{2} \right)$$

$$- P_I E \cdot \left(\frac{N_0^2 \eta^6}{120} + \frac{\eta^4}{12} + \frac{b_1^2 \eta^2}{2} + \frac{N_0 \eta^5}{20} - \frac{b_1 \eta^3}{3} - \frac{N_0 \eta^4}{12} b_1 \right) + E\eta + F, \quad (40)$$

where

$$b_1 = (M_0 + (N_0/2) + 1)$$

$$E = (B - BC_1 + BC_2 - BC_3 + BC_4 + BC_7 - BC_8 + BC_5 - BC_6)/(B+1)$$

$$F = (1 - C_1 + C_2 - BC_5 + BC_6 - BC_7 + BC_8 + BC_3 - BC_4)/(B+1).$$

ZONE-II

$$T_2 = P_{e2} aA' \left\{ \frac{N_0}{\sigma^2} \left(\frac{\eta^3}{6} + \frac{\eta^2}{2} \right) + \frac{b_2 e^{\eta\sigma}}{\sigma^2} \right\} - P_2 \cdot E \left(\frac{N_0^2}{\sigma^4} \frac{\eta^2}{2} + \frac{b_2^2 e^{2\eta\sigma}}{4} + \frac{2N_0 b_2 e^{\eta\sigma}}{\sigma^3} \right) + G\eta + H, \tag{41}$$

where

$$b_2 = (u_B - N_0/\sigma^2),$$

$$G = (1 - C_1 + C_2 - C_3 + C_4 - BC_5 + BC_6 + C_7 - C_8)/(B+1),$$

$$H = (1 - C_1 + C_2 - C_3 + C_4 - BC_5 + BC_6 - BC_7 + BC_8)/(B+1).$$

The constants in the above equations are defined below:

$$C_1 = \left(-\frac{N_0}{120} - \frac{1}{24} + \frac{b_1}{6} + \frac{u_B}{2} \right) P_{e1} aA',$$

$$C_2 = \left(\frac{N_0^2}{120} + \frac{1}{12} + \frac{b_1^2}{2} + \frac{N_0}{20} - \frac{b_1}{3} - \frac{N_0 b_1}{12} \right) P_1 \cdot E,$$

$$C_3 = \frac{b_2}{\sigma^2} P_{e2} aA'; C_4 = \left(\frac{b_2^2}{4} + \frac{2N_0 b_2}{\sigma^3} \right) P_2 \cdot E,$$

$$C_5 = \frac{b_2}{\sigma} P_{e2} aA'; C_6 = \left(\frac{b_2 \sigma}{2} + \frac{2N_0 b_2}{\sigma^2} \right) P_2 \cdot E,$$

$$C_7 = \frac{N_0}{3\sigma^2} P_{e2} aA'; C_8 = \frac{N_0}{2\sigma^4} P_2 \cdot E.$$

To evaluate the temperature distribution (presented in figures 5–8) we have considered the specific cases of thermal conductivities in porous media involving glass spheres and water ($B = 1{\cdot}25$, $k_1 = 0{\cdot}0014$, $k_2 = 0{\cdot}00176$) and steel spheres and water ($B = 62$, $k_1 = 0{\cdot}0014$, $k_2 = 0{\cdot}08696$). Figures 5 and 6 represent the temperature distribution for $P.E. = 0$ *i.e.* without dissipation. It is observed that increase in the conductivity value (B) has an appreciable effect on the temperature distribution and the profiles tend to be parabolic in nature. The same trend is observed by keeping B constant and varying the aspect ratio a. Figures 7 and 8 represent the temperature distribution in the presence of viscous dissipation. It is interesting to observe that the temperature profiles are completely reversed and the temperature at the nominal surface increases with increase in M_0, N_0 values. However, the influence of M_0 on the temperature rise is more compared to N_0. In other words, the surface tension is more dominant in increasing the temperature than buoyancy.

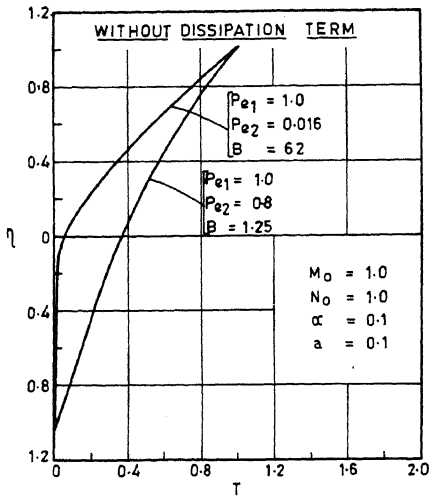


Figure 5. Temperature distribution *vs* η, without dissipation term.

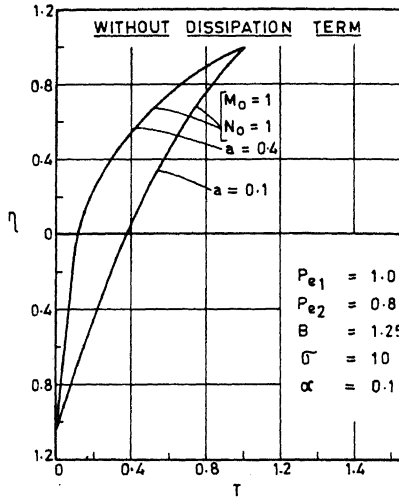


Figure 6. Temperature distribution *vs* η, with dissipation term.

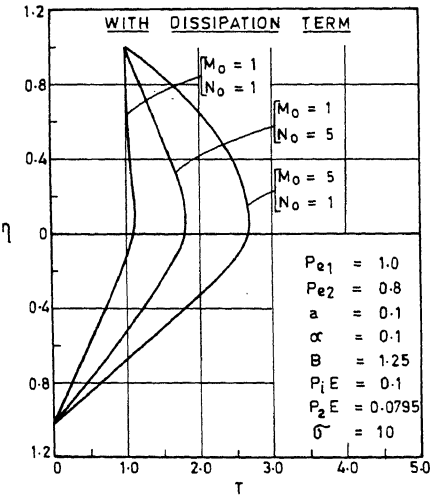


Figure 7. Temperature distribution *vs* η, with dissipation term.

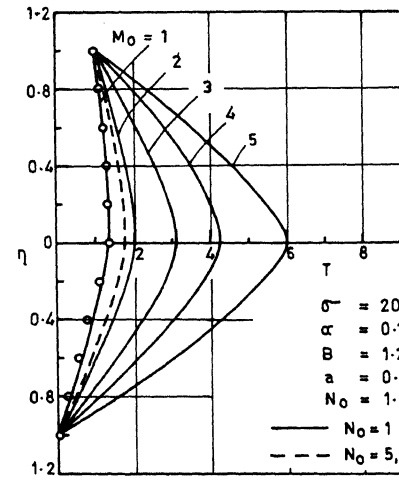


Figure 8. Temperature distribution *vs* η, dissipation term.

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Approximate analytical solutions for strong shocks with variable energy

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Abstract. Approximate analytical solutions are obtained for self-similar flows behind strong shocks with variable energy deposition or withdrawal at the wavefront in a perfect gas at rest with constant initial density. Numerical solutions are also obtained and the approximate solutions agree with these solutions. The effect of the adiabatic index on the solutions is investigated. The dependence of shock density ratio on the parameter characterizing the energy of the flow is studied. It is observed that the rate of deposition of energy at the wavefront decreases with increase of the parameter that specifies the total energy of the flow.

Keywords. Strong shocks; self-similar flows; variable energy; approximate analytical solutions.

1. Introduction

The study of blast waves with variable energy deposition at the wavefront is of interest in connection with the explosions produced by laser irradiation. With a constant power input, self-similar solutions were obtained for spherical shock [7]. The similarity solutions for blast waves with variable energy have also been studied [2, 3, 6]. A comprehensive parametric study of the solutions for blast waves with variable energy deposition at the wavefront was given by Barenblatt *et al* [1]. However, these studies are confined to obtain only numerical solutions.

In this paper we present approximate analytical solutions for self-similar flows with variable energy behind strong shock waves propagating in a medium at rest. Both deposition and withdrawal of energy at the wavefront are considered. The total energy of the flow behind the shock and the centre of explosition is assumed to be a function of shock radius obeying a power law. The medium considered here is a perfect gas with constant initial density. The basic assumption in obtaining the approximate solutions is that most of the mass of the flow is concentrated behind the shock-wave which is almost true for strong shocks. In particular, for self-similar flows, the method based upon this assumption, proposed by Laumbach and Probstein [4], is found to give simple solutions (see, for example, [5] and its references). Approximate solutions for cylindrical and spherical flows are obtained for the cases with addition and withdrawal of energy at the wavefront. The effect of the adiabatic index on spherical flows with constant power input is also investigated. The dependency of shock density ratio on the parameter characterizing the energy of the flow is studied. To compare with these approximate solutions, numerical solutions are also obtained for the specific values of the parameters characterizing the flows under study which are not investigated earlier. It is observed that there is a fairly good agreement between the numerical and approximate solutions. Also the approximate solutions require much less computational time as compared to numerical solutions.

2. Approximate analytical solutions

An approximate method developed by Laumbach and Probstein [4] is used for obtaining analytical solutions for the problem of self-similar flows of a perfect gas with variable energy deposition at the strong shock-wave propagating in a uniform medium at rest. The equations of continuity, momentum and adiabatic law of the flow in Lagrangian co-ordinates are of the form

$$\rho r^{\alpha} dr = \rho_0 r_0^{\alpha} dr_0, \tag{1}$$

$$\frac{\partial^2 r}{\partial t^2} + \frac{r^{\alpha}}{\rho_0 r_0^{\alpha}} \frac{\partial p}{\partial r_0} = 0, \tag{2}$$

$$p(r_0, t)/p_s(r_0) = [\rho(r_0, t)/\rho_s(r_0)]^{\gamma}, \tag{3}$$

where ρ_0 is the initial density of the perfect gas, p the pressure, ρ the density, t the time, r the Eulerian co-ordinate, r_0 the Lagrangian co-ordinate and γ is the ratio of specific heats. Suffix s represents quantities immediately behind the shock-wave and $\alpha = 0, 1, 2$ for plane, cylindrical and spherical flows respectively. Strong shock conditions can be written in the form

$$u_s = (1-\beta)\dot{r}_s, \tag{4}$$

$$\rho_s = \rho_0/\beta, \tag{5}$$

$$p_s = (1-\beta)\rho_0 \dot{r}_s^2, \tag{6}$$

$$\beta = [\gamma - (1 \pm 2(\gamma^2 - 1)q)^{1/2}]/(\gamma + 1), \quad q = Q/\dot{R}^2, \tag{7}$$

where the dot denotes differentiation with respect to time and Q is the energy per unit mass at the shock front. The negative sign before q represents addition of energy whereas positive sign represents withdrawal. It is assumed that Q varies such that q is always constant so that the problem is self-similar. Then the shock density ratio β is an unknown constant to be determined.

We assume that the total energy of the flow behind the shock varies as

$$E = E_0 r_s^n, \tag{8}$$

where E_0 and n are constants. Equation (2) can be written in the form

$$p(r_0, t) = p_s(r_s) + \int_{r_0}^{r_s} \frac{1}{r^{\alpha}} \partial^2 r/\partial t^2 \, \rho_0 r_0^{\alpha} dr_0. \tag{9}$$

The integral in (9) is evaluated approximately by replacing the integrand $(1/r^{\alpha})(\partial^2 r/\partial t^2)$ by its value at the shock, given in the Appendix. Then we get

$$p(r_0, t) = p_s(r_s) + \frac{\rho_0}{r_s^{\alpha}} \frac{\partial^2 r}{\partial t^2}\bigg|_s (r_s^{\alpha+1} - r_0^{\alpha+1})/(\alpha+1). \tag{10}$$

Using (8), the energy conservation equation in integral form may be written as

$$E_0 r_s^n/\sigma_{\alpha} = \int_0^{r_s} \frac{p}{(\gamma-1)} r^{\alpha} dr + \int_0^{r_s} 1/2 \left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^{\alpha} dr_0, \tag{11}$$

where $\sigma_{\alpha} = 2\pi\alpha + (\alpha - 1)(\alpha - 2)$. The integrals in (11) are evaluated approximately by replacing $p(r, t)$ by $p(0, t)$ and $\partial r/\partial t$ by $(\partial r/\partial t)|_s$. Using (10), we finally get a second order

differential equation from (11) as

$$\ddot{r}_s + A\dot{r}_s^2/r_s = Br_s^{(n-\alpha-2)}, \tag{12}$$

where

$$A = \frac{(\alpha+1)[2+(\gamma-1)(1-\beta)][\gamma(1-\beta)-\beta]+2\alpha\gamma\beta(1-\beta)}{2[2\beta+\gamma(1-\beta)]},$$

$$B = \frac{(\alpha+1)^2(\gamma-1)[\gamma(1-\beta)-\beta]}{(1-\beta)[2\beta+\gamma(1-\beta)]}\frac{E_0}{\rho_0\sigma_\alpha}.$$

Integration of (12) gives the shock propagation law as

$$r_s = Ct^\delta, \quad \delta = 2/(\alpha+3-n), \tag{13}$$

where

$$C = [2B/(n+2A-\alpha-1)\delta^2]^{\delta/2}.$$

Using (13), we get from (3) and (10) the approximate solutions for pressure and density as

$$p(r_0,t)/p_s(r_s) = F^\gamma(x), \tag{14}$$

$$\rho(r_0,t)/\rho_s(r_s) = F(x)x^b, \tag{15}$$

where

$$F(x) = [1+G(1-x^{\alpha+1})]^{1/\gamma},\ b = 2(1-\delta)/\gamma\delta, \tag{16a}$$

$$G = [2\beta+\gamma(1+\beta)](1-\delta)+\alpha\gamma\beta(1-\beta)\delta/(\alpha+1)[\beta-\gamma(1-\beta)]\delta, \tag{16b}$$

and $x = r_0/r_s$ is the reduced Lagrangian co-ordinate. From continuity equation, we get a relation between the reduced Eulerian co-ordinate $\lambda = r/r_s$ and x as

$$\lambda^{\alpha+1} = (\alpha+1)\beta\int_0^x \frac{x^{(\alpha-b)}}{F(x)}\,dx. \tag{17}$$

Using the condition at the shock that $\lambda = x = 1$, one gets a relation for obtaining β from (17) as

$$\frac{1}{\beta} = (\alpha+1)\int_0^1 \frac{x^{(\alpha-b)}}{F(x)}\,dx. \tag{18}$$

Differentiation of (17) w.r.t. time t gives approximate solution for velocity as

$$\frac{u(r_0,t)}{u_s(r_s)} = \frac{\beta}{(1-\beta)\lambda^\alpha}\int_0^x \frac{b(1+G)-G(b+(\alpha+1)/\gamma)x^{\alpha+1}}{F^{1+\gamma}(x)}x^{(\alpha-b)}dx. \tag{19}$$

For given values of n, α and γ the unknown parameter β is calculated by iterating (18). Then the approximate analytical solutions for pressure, density and velocity are obtained as functions of λ from (14), (15) and (19) respectively with the help of (17).

3. Numerical solutions

For comparison with the approximate solutions studied in § 2, we present here the

numerical solutions. Details of the method employed may be found in Wilson a Turcotte [7] and Barenblatt *et al* [1]. We take the similarity transformation as

$$u = \frac{r}{t} V(\lambda), \quad \rho = \rho_0 R(\lambda), \quad p = \rho_0 (r/t)^2 P(\lambda),$$

$$X = R/\gamma P, \quad \lambda = r/r_s. \tag{}$$

Using transformation (20), the basic equations of the flow behind the shock-wave n be written as

$$\frac{dX}{dV} = \frac{X}{(V-\delta)} \frac{(\gamma-1)[V(V-1)(V-\delta)X - (\alpha+1)(V-K)] + \{[(\alpha+1)(\gamma-1)+2]V-2\}[1-(V-\delta)^2 X]}{V(V-1)(V-\delta)X - (\alpha+1)(V-K)}, \tag{}$$

$$\frac{d\ln\lambda}{dV} = \frac{1-(V-\delta)^2 X}{V(V-1)(V-\delta)X + (\alpha+1)(K-V)}, \tag{}$$

$$\frac{d\ln R}{d\ln\lambda} = \frac{(\alpha+1)V}{(\delta-V)} - \frac{V(V-1)(V-\delta)X - (\alpha+1)(V-K)}{(V-\delta)[1-(V-\delta)^2 X]}, \tag{}$$

where

$$K = 2(1-\delta)/(\alpha+1)\gamma. \tag{}$$

Equations (20) together with the shock conditions give an expression for the Hugon curve as

$$X_s = [\gamma V_s(\delta - V_s)]^{-1}. \tag{}$$

The integral curve in the X, V phase plane can be obtained by integrating (21). Eac these curves are associated with a saddle point singularity $X = 0$ and $V = K$ wh corresponds to the centre of the wave. Using the Runge-Kutta method, (21 integrated from the singular point with the slope obtained from L'Hospital rule to point of intersection with the Hugoniot curve given by (25). This point of intersect gives the β value as the shock conditions are satisfied here. Once the integral curv known, it is straightforward to obtain the dependence of V and R on λ from (22) a (23) by using quadrature. The accuracy of solutions thus obtained can be checked fr the mass conservation integral

$$(\alpha+1)\int_0^1 R\lambda^\alpha \, d\lambda = 1. \tag{}$$

4. Results and discussion

For finding out the approximate solutions given by (14), (15) and (19) for the self-sim flows behind strong shocks propagating in a uniform medium at rest, we first find the values of β by iterating (18) for different values of n and γ for $\alpha = 1$ and $\alpha = 2$. Th values are also found numerically from (21) for the same cases and are listed in table is observed that the shock density ratio β increases with increase in n and γ. approximate solutions and also numerical solutions are plotted in figures 1 to 4 broken lines and solid lines respectively. One can find a very close agreement betw

Table 1. β values vs n for different values of α and γ.*

	β			
n	$\alpha = 2, \gamma = 1{\cdot}4$	$\alpha = 2, \gamma = 5/3$	$\alpha = 1, \gamma = 1{\cdot}4$	$\alpha = 1, \gamma = 5/3$
−0·5	0·086189 (0·087433)	0·174868 (0·171620)	0·050362 (0·052919)	0·133208 (0·136819)
0·0	0·168439 (0·166661)	0·254260 (0·250000)	0·164140 (0·166666)	0·248125 (0·250000)
0·5	0·258165 (0·255659)	0·338857 (0·334503)	0·296652 (0·298844)	0·370745 (0·373269)
1·0	0·353179 (0·350983)	0·426296 (0·422678)	0·431329 (0·434876)	0·492289 (0·496448)
1·5	0·417732 (0·416032)	0·513874 (0·511409)	0·555953 (0·557539)	0·606247 (0·607381)

*Values in brackets are from numerical solutions.

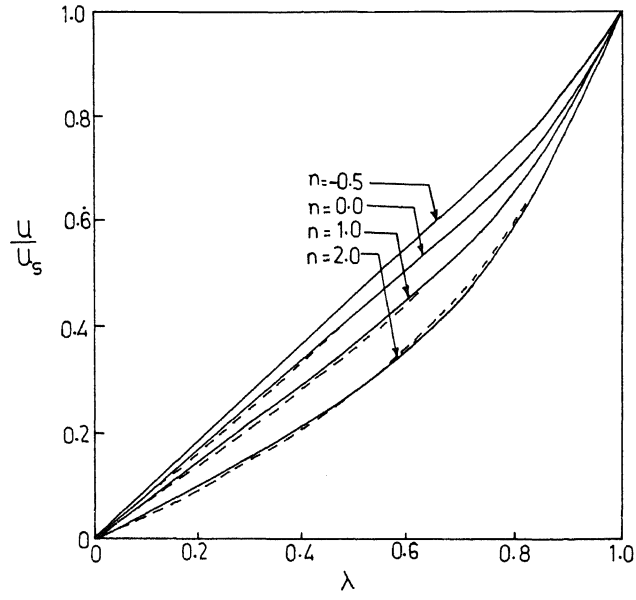

Figure 1. Velocity distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

these solutions. The effect of the adiabatic index γ on the approximate solutions is shown in figure 5 for $n = 5/3$ and $\alpha = 2$ which corresponds to the constant power input (Ranga Rao and Ramana [5]). In figure 6 the dependency of non-dimensional energy deposition at the wave-front q on n is shown from which one can conclude that the rate of deposition of energy at the front decreases as n increases.

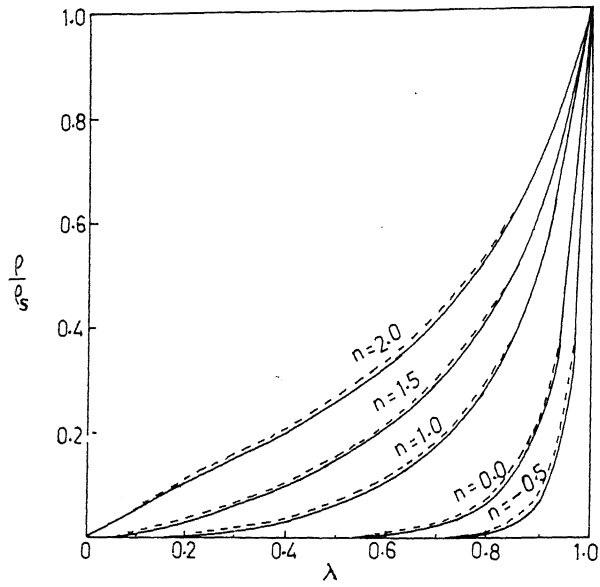

Figure 2. Density distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

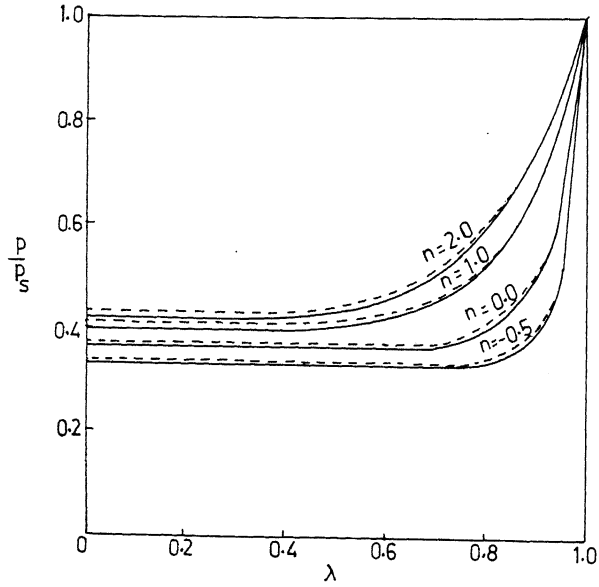

Figure 3. Pressure distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

Appendix

Taylor's expansion of the Eulerian co-ordinate $r(r_0, t)$ is written as

$$r(r_0, t) = r_s + \left.\frac{\partial r}{\partial r_0}\right|_s (r_0 - r_s) + 1/2 \left.\frac{\partial^2 r}{\partial r_0^2}\right|_s (r_0 - r_s)^2 + \ldots \tag{A.1}$$

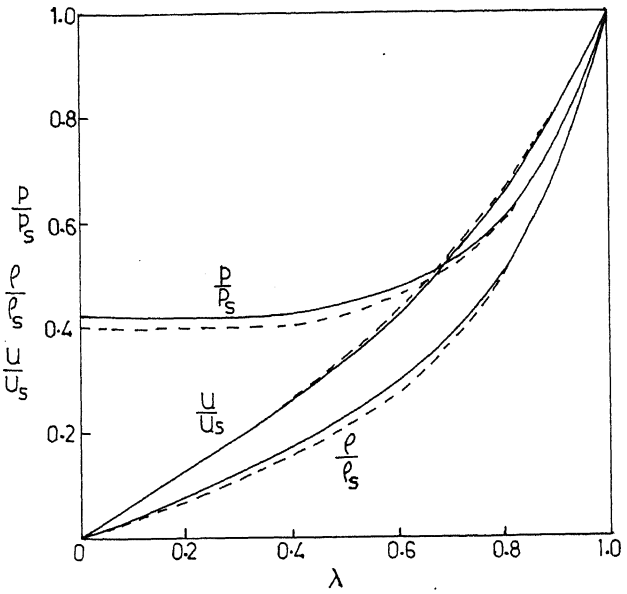

Figure 4. Flow distribution for $\gamma = 1{\cdot}4$, $\alpha = 1$, $n = 1$.

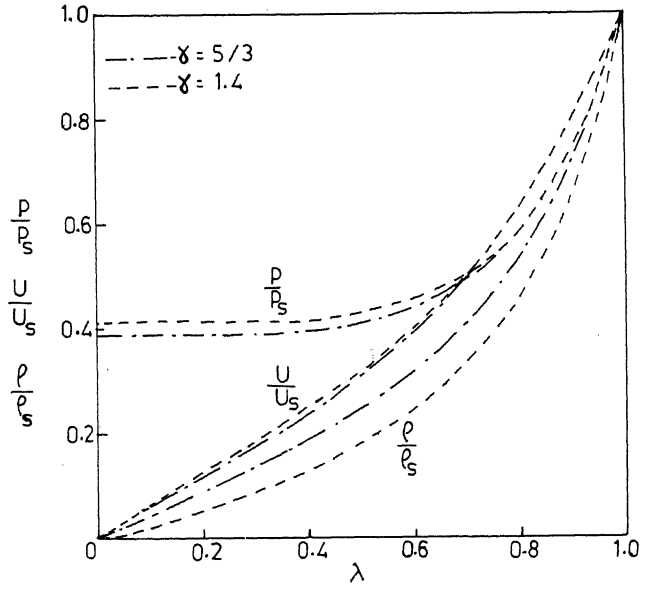

Figure 5. Approximate solutions for $n = 5/3$, $\alpha = 2$.

e equation of continuity (1) and the definition of β yield

$$\left.\frac{\partial r}{\partial r_0}\right|_s = \left.\frac{\rho_0 r_0^\alpha}{\rho r^\alpha}\right|_s = \beta, \tag{A.2}$$

$$\left.\frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\right|_s = \frac{\alpha(1-\beta)}{r_s} - \left.\frac{1}{\beta}\frac{\partial^2 r}{\partial r_0^2}\right|_s. \tag{A.3}$$

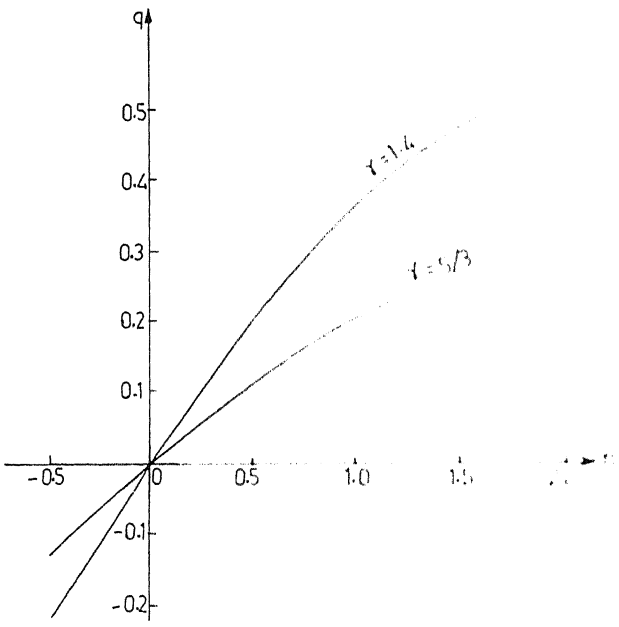

Figure 6. Non-dimensional energy deposition at the wave front q vs n for γ =

It follows from equations (A.1) and (A.2) that

$$\left.\frac{\partial r}{\partial t}\right|_s = (1-\beta)\dot{r}_s, \tag{}$$

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = (1-\beta)\ddot{r}_s + \left.\frac{\partial^2 r}{\partial r_0^2}\right|_s \dot{r}_s^2. \tag{}$$

Now equation (2) together with the shock condition (6) gives

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = -(1-\beta)\dot{r}_s^2 \left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)_s. \tag{}$$

Eliminating $(\partial^2 r/\partial t^2)|_s$ between (A.5) and (A.6) and also making use of (A.3), (5) and we get $(\partial^2 r/\partial r_0^2)|_s$ which after using in (A.5), we get finally

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = \frac{(1-\beta)}{\gamma(1-\beta)-\beta}\left[\{\gamma(1-\beta)+2\beta\}\ddot{r}_s + \alpha\gamma\beta(1-\beta)\left(r_s^2/r_s\right)\right]. \tag{}$$

Acknowledgement

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Flow between torsionally oscillating noncoaxial cylinders

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Abstract. The flow of an incompressible viscous fluid between two torsionally oscillating noncoaxial cylinders has been investigated. Closed form solutions for symmetric and first order asymmetric flow are obtained for the cases when the gap between the cylinders is finite. Solutions of the governing equations under the geometrical restriction of narrow gap are also presented. These solutions coincide with the solutions of the finite gap by incorporating in them the condition of narrow gap. The components of the force acting on the inner cylinder are calculated.

Keywords. Noncoaxial system; torsional oscillations; incompressible viscous fluid; closed form solution; symmetric flow.

1. Introduction

Two-dimensional flows between moving nearly coaxial cylinders are important in the design of control mechanisms for aircrafts and rockets and rheometers. Several authors have studied the viscous flow between two noncoaxial cylinders rotating about their axes which are parallel and situated slightly apart. Wood [8] in his study of the viscous flow between two noncoaxial rotating cylinders has obtained the exact solution of the governing Navier-Stokes equations assuming small eccentricity and discussed the asymptotic behaviour of these solutions for large Reynolds number. The conformal transformation technique has been used by Segel [5] to study the unsteady flow between noncoaxial cylinders when the outer cylinder is kept fixed and the inner cylinder rotates or vibrates about a slightly eccentric point. Separation in the flow between eccentric rotating cylinders has been investigated by Kamal [3]. Kulinski and Ostrach [4] also studied the flow between rotating cylinders with axes slightly apart in connection with their study of the flow phenomena in journal bearings. Flow between rotating eccentric cylinders has been considered by Abbot and Walters [1] in understanding some rheometrical flows. The steady flow between two noncoaxial rotating cylinders has been studied by Urban [6] by a simple approach using polar coordinate system. The principle of transfer of boundary conditions discussed by van Dyke [7] has been used to handle the conditions at the outer boundary which is not a coordinate curve.

In this paper, the primary oscillatory flow of a viscous incompressible fluid contained between two infinite cylinders with parallel axes situated slightly apart has been investigated. The solutions are obtained for the cases when the gap between the cylinders is finite and narrow. Components of the force acting on the inner cylinder are calculated. Further, it is observed from the profiles of transverse velocity distributions that the outer boundary condition is more clearly satisfied for large frequency parameter.

2. Formulation

Let O_1 and O_2 be the centres of the inner and outer cylinders with radii R_1 and R_2 respectively. O_1 is taken as the origin of the cylindrical polar coordinate system (r, θ, Z) with the Z-axis along the axis of the inner cylinder (figure 1). The distance O_1O_2 is the eccentricity e of the system. We define the non-dimensional parameters

$$\varepsilon = e/d, \ (0 < \varepsilon < 1), \text{ the eccentricity ratio,}$$

$$\delta = \frac{d}{R_0} (0 < \delta < 2), \text{ the gap ratio,}$$

where $d = R_2 - R_1$ and $R_0 = (R_1 + R_2)/2$. Using the cosine rule for the triangle $O_1 O_2 P$, the radial distance h which is the variable gap between the cylinders is obtained as

$$h(\theta) = -R_1 + \varepsilon d \cos\theta + R_2 \left(1 - \frac{4\varepsilon^2\delta^2}{(2+\delta)^2} \sin^2\theta\right)^{1/2}. \tag{1}$$

Consider the flow problem when the inner and outer cylinders execute rotatory oscillations with $\Omega_1 \exp(i\sigma t)$ and $\Omega_2 \exp(i\sigma t)$ respectively, where σ is the frequency of oscillations and Ω_1 and Ω_2 are the angular velocities of rotation. Assuming that there is no flow in the axial direction since the cylinders are infinite, the velocity and the pressure fields are functions of r, θ, and t only. The governing equations of the primary flow are

$$u_t = -\frac{1}{\rho'} p_r + \nu \left[u_{rr} + \frac{1}{r} u_r - \frac{u}{r^2} + \frac{1}{r^2} u_{\theta\theta} - \frac{2}{r^2} v_\theta \right] \tag{2}$$

$$v_t = -\frac{1}{r\rho'} p_\theta + \nu \left(v_{rr} + \frac{1}{r} v_r - \frac{v}{r^2} + \frac{1}{r^2} v_{\theta\theta} + \frac{2}{r^2} u_\theta \right). \tag{3}$$

and the continuity equation

$$u + r u_r + v_\theta = 0, \tag{4}$$

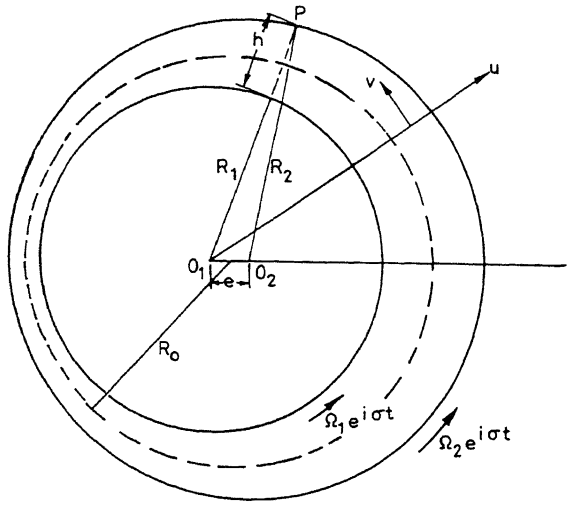

Figure 1. Torsionally oscillating non-coaxial cylinders.

where u is the radial velocity, v is the transverse velocity ρ' is the density, ν is the kinematic viscosity and p is the pressure. The subscripts denote the partial derivatives with respect to the corresponding variables and this notation is used throughout the paper. In a primary flow the nonlinear terms are completely omitted in the equations of motion, as their inclusion leads to a solution with second harmonic.

The boundary conditions are

$$\left.\begin{aligned} u &= 0 \\ v &= R_1\Omega_1 \exp(i\sigma t) \end{aligned}\right\} \quad \text{on } r = R_1, \tag{5}$$

$$\left.\begin{aligned} u &= -\Omega_2 \varepsilon d \sin\theta \exp(i\sigma t), \\ v &= R_2\Omega_2\{1-(\varepsilon d/R_2)^2 \sin^2\theta\}^{1/2} \exp(i\sigma t) \end{aligned}\right\} \quad \text{on } r = R_1 + h. \tag{6}$$

The boundary conditions (6) are derived making use of the fact that $r = R_1 + h$ is not a coordinate curve. Eliminating the pressure from (2) and (3) we get

$$u_t - rv_{tr} - v_t = \left[u_{rr\theta} - \frac{1}{r}u_{r\theta} + \frac{1}{r^2}u_\theta + \frac{1}{r^2}u_{\theta\theta\theta} \right.$$

$$\left. - \frac{1}{r^2}v_{\theta\theta} - 2v_{rr} - rv_{rrr} - \frac{v}{r^2} + \frac{1}{r}v_r - \frac{1}{r}v_{\theta\theta r} \right]. \tag{7}$$

In order to obtain the solution we expand u and v in powers of the small parameter ε in the form

$$u(r,\theta)\exp(i\sigma t) = \{u_0(r) + \varepsilon u_1(r,\theta) + \varepsilon^2 u_2(r,\theta) + \ldots\}\exp(i\sigma t), \tag{8}$$

$$v(r,\theta)\exp(i\sigma t) = \{v_0(r) + \varepsilon v_1(r,\theta) + \varepsilon^2 v_2(r,\theta) + \ldots\}\exp(i\sigma t). \tag{9}$$

Using (8) and (9) in (7) and (4) and equating different powers of ε on either side, we obtain various order equations which are solved to obtain the solutions.

3. Solution for finite gap

The symmetric flow due to torsionally oscillating coaxial cylinders is given by the zeroth order equation which is given in the non-dimensional form by

$$R^2 V_{0RR} + RV_{0R} + (-i\alpha^2 R^2 - 1)V_0 = 0, \tag{10}$$

where

$$\alpha^2 = \frac{\sigma}{\nu}R_1^2.$$

In writing (10) the following non-dimensional variables (capital letters) $UR_1\Omega_1 = u$, $VR_1\Omega_1 = v, R_1 R = r, T = \sigma t$, have been used. The corresponding boundary conditions are

$$V_0 = 1, \text{ at } R = 1,$$

$$V_0 = \mu/\eta \text{ at } R = 1/\eta, \tag{11}$$

where

$$\eta = R_1/R_2 \text{ and } \mu = \Omega_2/\Omega_1. \tag{12}$$

The solution of (10) satisfying (11) is

$$V_0 = \frac{1}{D}[Y_1(\rho_2)J_1(\rho) - J_1(\rho_2)Y_1(\rho) + \mu/\eta\{J_1(\rho_1)Y_1(\rho) - Y_1(\rho_1)J_1(\rho)\}], \tag{13}$$

where

$$\lambda = \exp\left(-i\frac{\pi}{4}\right)\alpha, \quad \rho_1 = \lambda, \quad \rho = \lambda R, \quad \rho_2 = \lambda/\eta$$

and

$$D = J_1(\rho_1)Y_1(\rho_2) - J_1(\rho_2)Y_1(\rho_1), \tag{14}$$

J_n, Y_n being Bessel functions of order n. It is easily seen that the expression for V_0 given in (13) coincides with the one given by Urban [6], for steady flow, in the limit $\lambda \to 0$.

The equation governing the asymmetric flow is

$$i\alpha^2(U_{1\theta} - RV_{1R} - V_1) = \left[U_{1RR\theta} \times \frac{1}{R^2}U_{1\theta\theta\theta} + \frac{2}{R^2}U_{1\theta} + \frac{1}{R}V_{1R} - 2V_{1RR} - RV_{1RRR} - \frac{1}{R}V_{1\theta\theta\theta} - \frac{V_1}{R^2}\right]. \tag{15}$$

In order to obtain the first order boundary conditions, we adopt the following procedure. In view of (1) it is noted that the perturbation parameter ε appears implicitly in the radial coordinate in the expressions for u_1 and v_1 as well as explicitly. Hence it is not possible to equate the various powers of ε directly. This difficulty is overcome by expanding u_1 and v_1 in Taylor's series about r equal to R_2, which gives the explicit dependence of u_1 and v_1 on ε. This method is known as 'the transfer of boundary conditions', see van Dyke [7]. Now, the boundary conditions (5) and (6) reduce at first order in ε to

$$\left.\begin{aligned} U_1 &= 0 \\ V_1 &= 0 \end{aligned}\right\} \quad \text{on } R = 1, \tag{16}$$

$$\left.\begin{aligned} U_1 &= (-\mu/R_1)d\sin\theta \\ V_1 &= (-K_0 d/R_1)\cos\theta \end{aligned}\right\} \quad \text{on } R = 1/\eta, \tag{17}$$

where

$$K_0 = (V_{0R})_R = 1/\eta. \tag{18}$$

Equation (15) admits a solution of the form

$$U_1 = \tilde{U}(R)i\exp(i\theta), \; V_1 = -(R\tilde{U})_R\exp(i\theta). \tag{19}$$

Substituting (19) in (15) we get

$$\tilde{U}_{RRRR} + \frac{6}{R}\tilde{U}_{RRR} + \frac{3}{R^2}\tilde{U}_{RR} - \frac{3}{R^3}\tilde{U}_R - i\alpha^2(\tilde{U}_{RR} + \frac{3}{R}\tilde{U}_R) = 0. \tag{20}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} \tilde{U} &= 0, \\ \tilde{U}_R &= 0 \end{aligned}\right\} \quad \text{at } R = 1, \tag{21}$$

$$\left.\begin{aligned}\tilde{U} &= \mu d/R_1 \\ \tilde{U}_R &= \eta d\,(K_0 - \mu)/R_1\end{aligned}\right\} \qquad \text{at } R = 1/\eta. \tag{22}$$

The solution of (20) satisfying (21) and (22) is

$$\tilde{U} = AI_1(\rho) + BI_2(\rho), \tag{23}$$

where

$$A = \frac{d}{R_1\Delta}\{\mu I_4(\rho_2) - K_0 I_2(\rho_2)\},$$

$$B = \frac{d}{R_1\Delta}\{K_0 I_1(\rho_2) - \mu I_3(\rho_2)\},$$

$$I_1(\rho) = \{J_0(\rho_1) - J_0(\rho) - J_2(\rho) + \rho_1^2 J_2(\rho_1)/\rho^2],$$

$$I_2(\rho) = [Y_0(\rho_1) - Y_0(\rho) - Y_2(\rho) + \rho_1^2 Y_2(\rho_1)/\rho^2],$$

$$I_3(\rho) = [J_0(\rho_1) - J_0(\rho) + J_2(\rho) - \rho_1^2 J_2(\rho_1)/\rho^2],$$

$$I_4(\rho) = [Y_0(\rho_1) - Y_0(\rho) + Y_2(\rho) - \rho_1^2 Y_2(\rho_1)/\rho^2]$$

and

$$\Delta = I_1(\rho_2)I_4(\rho_2) - I_3(\rho_2)I_2(\rho_2).$$

The radial and transverse velocities are given by

$$U_1 = \frac{d}{R_1\Delta}[\mu\{I_4(\rho_2)I_1(\rho) - I_3(\rho_2)I_2(\rho)\} + K_0\{I_1(\rho_2)I_2(\rho) - I_2(\rho_2)I_1(\rho)\}]\, i \exp(i\theta), \tag{24}$$

$$V_1 = \frac{d}{R_1\Delta}[\mu\{I_3(\rho_2)I_4(\rho) - I_4(\rho_2)I_3(\rho)\} - K_0\{I_1(\rho_2)I_4(\rho) - I_2(\rho_2)I_3(\rho)\}] \exp(i\theta). \tag{25}$$

It is not possible to derive the asymmetric steady flow results discussed by Urban [6] from our results (24) and (25) as the nonlinear terms are neglected completely in obtaining them. Segel [5] has obtained results similar to (24) and (25) in the study of the oscillatory flow between two circular cylinders where the inner cylinder remains fixed and the centre of the outer cylinder vibrates causing the asymmetry.

4. Solution for narrow gap

In this section some useful and simple results are derived by considering an extra restriction namely, the gap between the cylinders is very small in addition to the small eccentricity. First the geometrical condition of narrow gap is introduced into the governing equations and boundary conditions from which the solutions are presented. We introduce a non-dimensional independent variable X by

$$r = R_0(1 + \delta X), \tag{26}$$

where δ, the gap ratio defined earlier is very small. The new independent variable has the

range $-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}$ and any point in the fluid domain is prescribed by X and θ.

Introducing (26) in (10) one obtains

$$i\beta^2\delta^2 V_0 = \left[V_{0XX} + \frac{\delta}{(1+\delta X)} V_{0X} - \frac{\delta^2 V_0}{(1+\delta X)^2} \right], \tag{27}$$

where

$$\beta^2 = R_0^2 \frac{\sigma}{\nu}. \tag{28}$$

Here two interesting cases arise:

(i) $\beta^2\delta^2 \to 0$ as $\delta \to 0$ implying that the frequency parameter β is small,

(ii) $\beta^2\delta^2 = (\sigma/\nu)d^2 = i\gamma^2$, (finite) as $\delta \to 0$ which implies that β is large enough to make $\beta\delta$ finite in the limit $\delta \to 0$. Taking the limit as $\delta \to 0$ in (27) in case (i) we get

$$V_{0XX} = 0. \tag{29}$$

The corresponding boundary conditions are

$$V_0 = 1 \text{ at } x = -\tfrac{1}{2},$$
$$V_0 = \mu \text{ at } x = \tfrac{1}{2}. \tag{30}$$

The solution of (29) satisfying (30) is

$$V_0 = \left[\frac{1+\mu}{2} - (1-\mu)X \right]. \tag{31}$$

Taking the limit as $\delta \to 0$ in (27) in case (ii) one obtains

$$V_{0XX} + \gamma^2 V_0 = 0. \tag{32}$$

The solution of (32) satisfying (30) is

$$V_0 = \frac{1}{\sin\gamma} [\mu \sin\gamma(x+\tfrac{1}{2}) - \sin\gamma(x-\tfrac{1}{2})]. \tag{33}$$

It is easily seen that (33) reduces to (31) as $\gamma \to 0$. We observe that the solution given in (33) for the narrow gap is also derivable from the finite gap solution (13). This is achieved in the following way: Replace the Bessel function in (13) by their asymptotic series for large frequency parameter, change the independent variable R to X as defined in (26) and then take the limit as $\delta \to 0$, keeping $\beta\delta$ finite.

Introducing the narrow gap approximation in (19)–(21) for case (i) leads to

$$\tilde{U}_{XXXX} = 0, \tag{34}$$
$$\tilde{U}(X) = 0, \quad \tilde{U}_X = 0 \text{ at } x = -\tfrac{1}{2}, \tag{35}$$
$$\tilde{U}(X) = \mu \quad \tilde{U}_X = (\mu - 1) \text{ at } x = \tfrac{1}{2}. \tag{36}$$

The solution for (34) satisfying (35) and (36) is

$$\tilde{U} = \frac{3\mu+1}{8} + \frac{5\mu+1}{4} X + \frac{\mu-1}{2} X^2 - (\mu+1)X^3. \tag{37}$$

The first order radial and transverse velocities are obtained as

$$U_1(X,\theta) = \left[\frac{3\mu+1}{8} + \frac{5\mu+1}{4}X + \frac{\mu-1}{2}X^2(\mu+1)X^3\right] i\exp(i\theta), \tag{38}$$

$$V_1(X,\theta) = -\left[\frac{5\mu+1}{4} + (\mu-1)X - 3(\mu+1)X^2\right]\exp(i\theta). \tag{39}$$

By taking the real parts of the results given in (31), (38) and (39) we get the results of the steady flow given by Urban [6]. In our problem these are to be multiplied with $\exp(iT)$ as given in (8) and (9) and then the real parts are to be taken. Thus in case (i), the conclusions drawn by Urban [6] holds good in our problem also except for the time dependence. The governing equations and boundary conditions in case (ii) become

$$\tilde{U}_{XXXX} + \gamma^2\tilde{U}_{XX} = 0, \tag{40}$$

$$\tilde{U} = 0, \quad \tilde{U}_X = 0 \text{ at } x = -\tfrac{1}{2}, \tag{41}$$

$$\tilde{U} = \mu, \quad \tilde{U}_X = \frac{\gamma}{\sin\gamma}(\mu\cos\gamma - 1) \text{ at } x = \tfrac{1}{2}. \tag{42}$$

The solution of (40) satisfying (41) and (42) is obtained and the velocities are given by

$$U_1(X,\theta) = [C_1 + C_2X + C_3\sin\gamma(X+\tfrac{1}{2}) + C_4\sin\gamma(X-\tfrac{1}{2})]i\exp(i\theta), \tag{43}$$

$$V_1(X,\theta) = -[C_2 + C_3\gamma\cos\gamma(X+\tfrac{1}{2}) + C_4\gamma\cos\gamma(X-\tfrac{1}{2})]\exp(i\theta). \tag{44}$$

where

$$C_1 = d_1/d^*,\ C_2 = d_2/d^*,\ C_3 = d_3/d^*,\ C_4 = d_4/d^*,$$

$$d_1 = \gamma/2[(1-\mu)(\gamma+\gamma\cos\gamma - 2\sin\gamma) - \gamma\mu\cos^2\gamma],$$

$$d_2 = \gamma^2[(1+\mu)(\cos\gamma - 1)],$$

$$d_3 = \gamma/\sin\gamma[(1+\mu)\sin\gamma + \gamma\cos\gamma(\mu\cos\gamma - 1) - 2\mu\sin\gamma\cos\gamma],$$

$$d_4 = \gamma/\sin\gamma[\mu\sin\gamma - \sin\gamma - \gamma(\mu\cos\gamma - 1)],$$

$$d^* = \gamma/\sin\gamma(2 - 2\cos\gamma - \gamma\sin\gamma).$$

The solutions given in (43) and (44) can be obtained from (24) and (25) following the same procedure explained for zeroth order velocity.

5. Discussion

The physical quantities of interest in this problem are the forces acting on the cylinders. If X' and Y' denote the components of force on the inner cylinder in the x and y directions respectively, then we have for a cylinder of length h, Abbot *et al* [2],

$$X' = R_1 h\int_0^{2\pi}(P_{rr}\cos\theta - P_{r\theta}\sin\theta)\,d\theta, \tag{45}$$

$$Y' = R_1 h\int_0^{2\pi}(P_{rr}\sin\theta + P_{r\theta}\cos\theta)\,d\theta, \tag{46}$$

where P_{rr} and $P_{r\theta}$ are the components of stress tensor. Substituting the expressions for P_{rr} and $P_{r\theta}$ in terms of non-dimensional velocities, we get

$$X' = i\pi R_1 h v \rho' \Omega_1 \varepsilon \left\{ R \frac{d^2 \tilde{U}}{dR^2} + 3 \frac{d\tilde{U}}{dR} \right\}_{R=1} \exp(iT), \tag{47}$$

$$Y' = iX. \tag{48}$$

where the real parts on the right sides of (47) and (48) only are relevant in a physical problem. Using $\tilde{U}$ of finite gap given by (23) in (47), the force in the x-direction is

$$X' = 2\pi i v \rho' R_1 h \Omega_1 \varepsilon \lambda [A J_1(\lambda) + B Y_1(\lambda)] \exp(iT). \tag{49}$$

From (47) using the narrow gap expression for $\tilde{U}$ in case (ii), we get

$$X' = i\pi R_1 v \rho' h \Omega_1 \varepsilon / \delta [d_4 \gamma^2 \sin\gamma / d^*] \exp(iT). \tag{50}$$

Employing in (47) the narrow gap expression for $\tilde{U}$ given in (37) for case (i) and collecting the real parts we get

$$X' = -2\pi R_1 v \rho' h \Omega_1 \varepsilon / \delta (2\mu + 1) \sin \sigma t, \tag{51}$$

$$Y' = -2\pi R_1 v \rho' h \Omega_1 \varepsilon / \delta (2\mu + 1) \cos \sigma t. \tag{52}$$

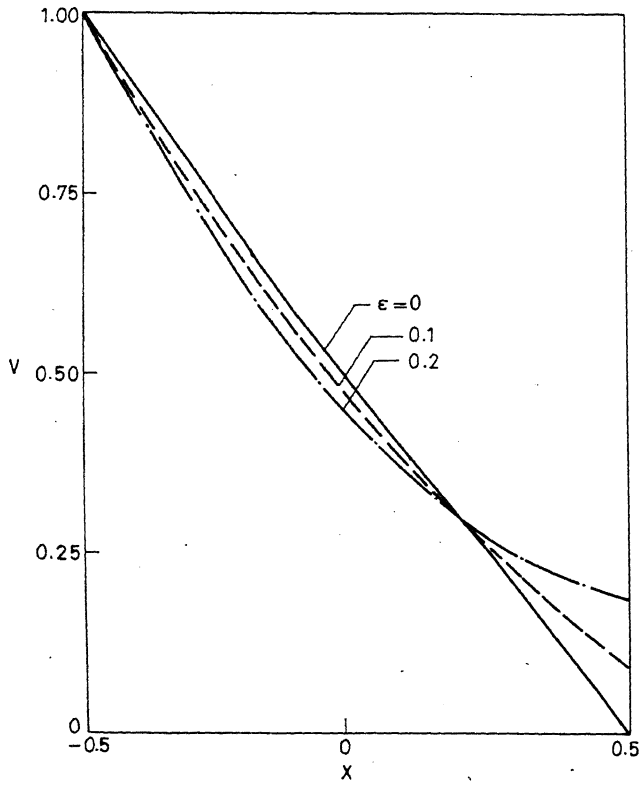

Figure 2. Transverse velocity profiles for $\gamma = 1 - i$, $\theta = 0°$ and $T = 0$.

From (51) and (52) we observe that in the limit $\sigma \to 0$, Y' only exists. This means that for steady, rotating non-coaxial cylinders, the force on the inner cylinder is always perpendicular to the line of centres.

The small gap transverse velocity plays a significant role in a stability analysis and has been discussed by Urban [6] for the steady problem. As our results in case (i) at time $t = 0$ are the same as the results of Urban [6], we discuss only the narrow gap transverse velocity profiles in case (ii). Figures 2 and 3 depict the transverse velocity profiles when the outer cylinder is at rest ($\mu = 0$) for various values of eccentricity ratio ε and the frequency parameter γ with values $1 - i$ and $(1{\cdot}5) - (1{\cdot}5i)$ respectively. The symmetric velocity corresponding to $\varepsilon = 0$ is not a straight line, as seen from (33) and the figures, unlike in case (ii). It is seen that the outer boundary condition is not well satisfied when we include the first order solution. The error in the transverse velocity at the outer boundary is positive, increases with the increase of the eccentricity ratio ε. Comparing figures 2 and 3 we see that the outer boundary condition is more closely satisfied when the value of the frequency parameter is large. The transverse velocity profiles for $\theta = 180°$ and the other parameters remaining the same as in the previous case are shown in figures 4 and 5. Here the error in the transverse velocity at the outer boundary is negative. The error at the outer boundary is found to decrease with increase of the frequency parameter at both the locations $\theta = 0°$ and $\theta = 180°$. Thus for given ε small,

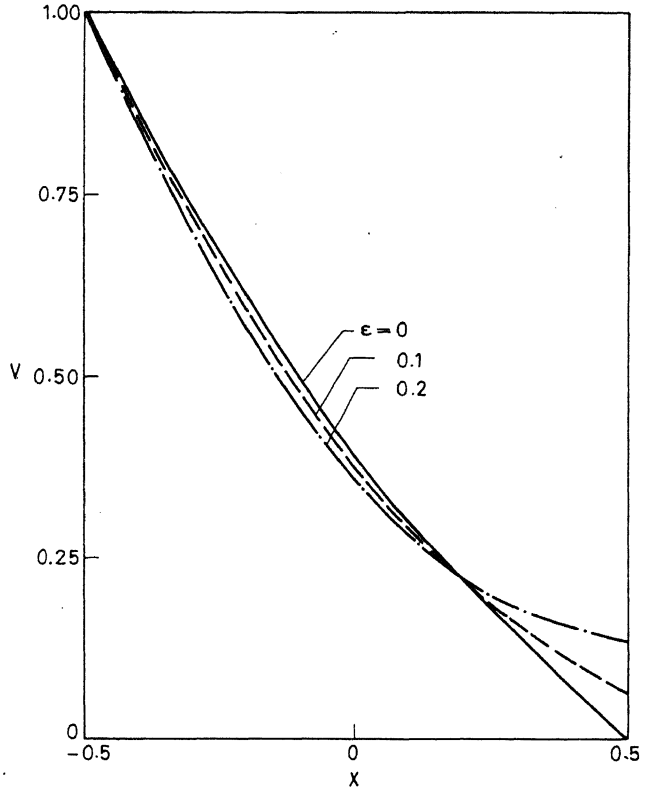

Figure 3. Transverse velocity profiles for $\gamma = 1{\cdot}5 - (1{\cdot}5)i$, $\theta = 0°$ and $T = 0$.

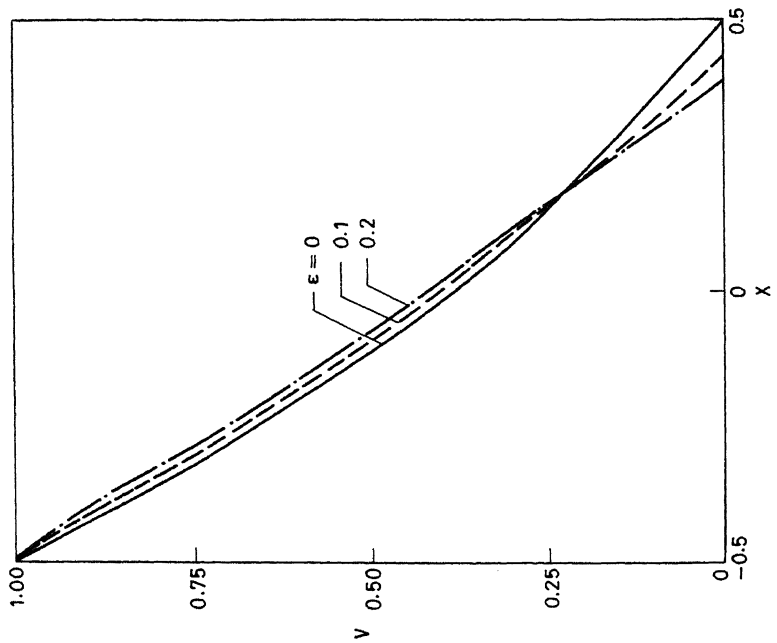

Figure 5. Transverse velocity profiles for $\gamma = 1{\cdot}5 - i(1{\cdot}5)$, $\theta = 180°$ and $T = 0$.

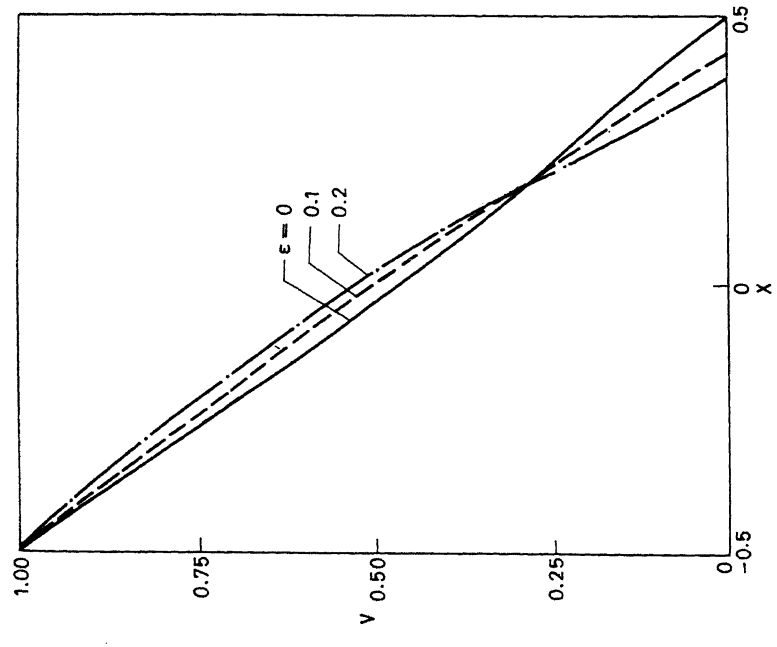

Figure 4. Transverse velocity profiles for $\gamma = 1 - i$, $\theta = 180°$ and $T = 0$.

the boundary condition at the outer boundary is satisfied more closely for sufficiently large frequency parameter.

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On diffraction of SH-waves by cuts in nonhomogeneous solids

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Abstract. The diffraction of SH-waves by an infinite periodic system of cuts in an infinite medium possessing nonhomogeneity has been studied. Assuming that shear modulus and density vary, the problem of diffraction of SH-waves by the periodic system of cuts depends on the solution of dual series equations which ultimately reduces to the solution of an infinite system of algebraic equations.

Keywords. Diffraction of elastic waves; cuts and cracks; stress intensity factor; SH-waves.

1. Introduction

In most of the problems of diffraction of elastic waves by cuts or cracks the solution has been sought for low or high frequency approximations. Sommerfield [8], Sih and Loeber [5, 6], Mal [2, 3], Thau and Lu [10] and many others investigated such problems. Parton and Morozov [4] presented the problem of diffraction of shear waves by a periodic system of cracks lying in an infinite homogeneous elastic medium.

Studies on the diffraction by cuts and cracks in nonhomogeneous elastic medium have been scanty. Singh and Dhaliwal [7] considered a static problem of crack in a nonhomogeneous medium in which shear modulus decays exponentially on both sides of the crack. To understand the effect of nonhomogeneity on diffraction of SH-type wave by a periodic system of cuts in an infinite medium we consider the simple case of exponentially varying shear modulus and density on the two sides of the cuts such that their ratio always remains the same. The resulting diffraction problem has been solved and the stress intensity factor (SIF) has been calculated. The effect of nonhomogeneity on SIF is more prominent for smaller wave numbers than for larger ones, as shown by the numerical results.

2. Formulation of the problem

There is an infinite row of collinear cracks of equal length which are equally spaced on a line taken as the x_1-axis. The length of the crack is $2l$ and the distance between the centres of the crack is $2L$.

We suppose that displacement $(0, 0, W_1)$ is produced by SH-waves coming from infinity (*i.e.* positive x_2-side) given by

$$W_1 = w^{(1)}(x_2) \exp(-i\Omega t) \tag{1}$$

where Ω in the time dependent factor in (1) represents the circular frequency.

Equation of motion in the absence of body forces is

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{2}$$

where σ_{ij} is the stress tensor $(i,j = 1,2,3)$; u_i are the components of the displacem vector $(i,j = 1,2,3)$; ρ is the density of the medium; comma denotes covar differentiation and dot denotes time derivatives.

The non-zero components of stresses are clearly

$$\sigma_{23} = \sigma_{32} = \mu \frac{\partial w^{(1)}}{\partial x_2} \exp(-i\Omega t).$$

We assume the material nonhomogeneity to be of the form

$$\mu = \mu_0 \exp(-\delta x_2); \quad \rho = \rho_0 \exp(-\delta x_2)$$

The only nontrivial equation of the system (2) is

$$\frac{d^2}{dx_2^2} w^{(1)} - \delta \frac{d}{dx_2} w^{(1)} + \frac{\Omega^2}{c^2} w^{(1)} = 0$$

where $c^2 = (\mu_0/\rho_0)^{1/2}$.

Hence the shear wave solution of (5) is given by, on replacing x_1, x_2, x_3 by x, respectively,

$$W_1 = w_0 \exp\left[\left(\frac{\delta}{2} - i(\lambda^2 - \delta^2/4)^{1/2}\right) y - i\Omega t\right]$$

where $\lambda^2 = \Omega^2/c^2$

The disturbance field is taken to be of the form $(0, 0, W_2)$ where

$$W_2 = w^{(2)}(x, y) \exp(-i\Omega t)$$

and leads to the equation

$$\frac{\partial^2}{\partial x^2} w^{(2)} + \frac{\partial^2}{\partial y^2} w^{(2)} - \delta \frac{\partial}{\partial y} w^{(2)} = -\lambda^2 w^{(2)}.$$

Defining the nondimensional quantities

$$\xi = x(\pi/L) \text{ and } \eta = y(\pi/L).$$

we assume the solution, periodic in ξ, to be of the form

$$w^{(2)} = \cos n\xi \cdot F(\eta).$$

Remembering that W_2 should vanish as $\eta \to \pm\infty$ it can be shown easily t considering (10) and (8) we have

$$W_2 = \sum_{n=1}^{\infty} A_n \cos n\xi \exp\left[\left(\frac{\delta L}{2\pi} - (n^2 - \beta^2)^{1/2}\right)\eta - i\Omega t\right], \quad \eta > 0,$$

$$= \sum_{n=1}^{\infty} B_n \cos n\xi \exp\left[\left(\frac{\delta L}{2\pi} + (n^2 - \beta^2)^{1/2}\right)\eta - i\Omega t\right], \quad \eta < 0,$$

where

$$\beta^2 = \alpha^2 - \delta^2 L^2/4\pi^2 \quad \text{and} \quad \alpha^2 = \Omega^2 L^2/c^2\pi^2.$$

The final displacement on superposing the two solutions becomes

$$W = W_1 + W_2$$

where W_1 and W_2 are given respectively by (6) and (11).

3. Boundary conditions of the problem

The boundary conditions to the problem can be formulated as

(i) $\sigma_{yz}|_{y=0} = 0$ in $0 < x \leqslant l$ (14)

where $\sigma_{yz} = \sigma_{yz}^{(1)} + \sigma_{yz}^{(2)}$

with $\sigma_{yz}^{(1)} = \mu \dfrac{\partial w_1}{\partial y}$ and $\sigma_{yz}^{(2)} = \mu \dfrac{\partial W_2}{\partial y}$

and (ii) W is continuous on $y = 0$ in $l < x \leqslant L$. (15)

4. Solution to the problem

We introduce a new constant C_n defined by

$$A_n = (\delta L/2\pi + (n^2 - \beta^2)^{1/2}) C_n/(n^2 - \alpha^2)^{1/2}$$

$$B_n = (\delta L/2\pi - (n^2 - \beta^2)^{1/2}) C_n/(n^2 - \alpha^2)^{1/2}$$

so that (14) gives

$$\sum_{n=1}^{\infty} (n^2 - \alpha^2)^{1/2} C_n \cos n\xi = Lp_0/\pi\mu_0, \; 0 < \xi \leqslant \xi_0 \tag{16}$$

where $p_0 = \mu_0 w_0 (\delta/2 - i(\lambda^2 - \delta^2/4)^{1/2})$, $\xi_0 = \pi l/L$ and the condition (15) leads to the equation.

$$\sum_{n=1}^{\infty} C_n \left(\frac{n_2^2 - \beta^2}{n^2 - \alpha^2} \right)^{1/2} \cos n\xi = 0, \quad \xi_0 < \xi \leqslant \pi. \tag{17}$$

Taking

$$C_n \left(\frac{n^2 - \beta^2}{n^2 - \alpha^2} \right)^{1/2} = D_n, \tag{18}$$

and

$$\frac{n^2 - \alpha^2}{(n^2 - \beta^2)^{1/2}} = n - \Phi(n) \tag{19}$$

we have from (16) and (17) the dual series equations

$$\sum_{n=1}^{\infty} [n - \Phi(n)] D_n \cos n\xi = Lp_0/\pi\mu_0, \quad 0 < \xi \leqslant \xi_0 \tag{20}$$

$$\sum_{n=1}^{\infty} D_n \cos n\xi = 0, \quad \xi_0 < \xi \leqslant \pi \tag{21}$$

where $\Phi(n)$ is given by

$$\Phi(n) = [n - (n^2 - \beta^2)^{1/2}] + \frac{\delta^2 L^2}{4\pi^2 (n^2 - \beta^2)^{1/2}}.$$

We thus reduce the problem to the determination of D_n from the dual series equations (20) and (21).

Following [4] the solution to the dual series equation (20) and (21) can be found as

$$D_n = \frac{1}{\sqrt{2}} \int_0^{\xi_0} g(t) y_n(\cos t)\, dt$$

where $y_n(\cos t) = P_n(\cos t) + P_{n-1}(\cos t)$ (22)

$P_n(x)$ being Legendre polynomials the unknown function $g(t)$ appearing in (21) satisfies the integral equation

$$\int_0^{\xi_0} \frac{g(t)\, dt}{(\cos t - \cos \xi)^{1/2}} = \psi(\xi) \tag{23}$$

where

$$\psi(\xi) = \sec \frac{\xi}{2} \left[\int_0^{\xi_0} g(t) \left(\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\Phi(n)}{n} y_n(\cos t) \sin n\xi \right) dt \right] + \frac{Lp_0}{\pi \mu_0} \xi. \tag{24}$$

Following Srivastava [9] the integral equation (24) is solved as

$$g(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{\sin \xi\, \psi(\xi)\, d\xi}{(\cos \xi - \cos t)^{1/2}}. \tag{25}$$

Replacing $\psi(\xi)$ by (24) in (25) and simplifying, the unknown function $g(t)$ is found to be of the form

$$g(t) = \tfrac{1}{2} \tan \frac{t}{2} \int_0^{\xi_0} g(\tau) \left[\sum_{n=1}^{\infty} \Phi(n) y_n(\cos t) y_n(\cos \tau) \right] d\tau + \frac{Lp_0}{\pi \mu_0} \sqrt{2} \tan t/2.$$

Denoting

$$\int_0^{\xi_0} g(\tau) y_m(\cos \tau)\, d\tau = a_m$$

$g(t)$ can be expressed as

$$g(t) = \tfrac{1}{2} \tan \frac{t}{2} \sum_{n=1}^{\infty} a_n \Phi(n) y_n(\cos t) + \frac{Lp_0}{\pi \mu_0} \sqrt{2} \tan t/2. \tag{26}$$

Now using the orthogonality condition of the well-known y_n functions one may at once derive from (26) the following system of algebraic equations

$$a_m = \sum_{n=1}^{\infty} a_n d_{mn} + \frac{Lp_0}{\pi \mu_0} \sqrt{2} b_m \tag{27}$$

where

$$d_{mn} = \tfrac{1}{2} \Phi(n) I_{mn}(\xi_0), \tag{28}$$

with

$$I_{mn}(\xi_0) = \int_0^{\xi_0} \tan \frac{t}{2}\, y_m(\cos t)\, y_n(\cos t)\, dt \tag{29}$$

and

$$b_m = \int_0^{\xi_0} \tan \frac{t}{2}\, y_m(\cos t)\, dt. \tag{30}$$

The solution to the infinite system of equations (27) determines the constants a_n appearing in (26) (Kantarovich and Krylov [1]).

We now calculate the stress $\sigma_{yz}|_{y=0}$. From (6) and (11) via (18) and (19) we obtain

$$\sigma_{yz}|_{y=0} = \mu_0 p_0 \exp(-i\Omega t) - \frac{\pi\mu_0}{L} \sum_{n=1}^{\infty} \{[n-\Phi(n)] D_n \cos n\xi \exp(-i\Omega t)\}, \quad 0 < \xi \leqslant \xi_0$$

which is rewritten as

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\mu_0 p_0 - \frac{\pi\mu_0}{L}\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\sum_{n=1}^{\infty} D_n \sin n\xi - \sum_{n=1}^{\infty}\frac{\Phi(n)}{n} D_n \sin n\xi\right\}\right].$$

On substituting D_n from (22) in above and using the relation

$$\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} y_n(\cos t) \sin n\xi = \frac{\cos\frac{1}{2}\xi H(\xi - t)}{(\cos t - \cos\xi)^{1/2}},$$

$H(\xi - t)$ being the Heaviside unit step function, we obtain

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\mu_0 p_0 - \frac{\pi\mu_0}{L}\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\cos\tfrac{1}{2}\xi \int_0^{\xi_0} \frac{g(t)\,\mathrm{d}t}{(\cos t - \cos\xi)^{1/2}} + \frac{1}{\sqrt{2}}\int_0^{\xi_0} g(t)\left(\sum_{n=1}^{\infty}\frac{\Phi(n)}{n} y_n(\cos t)\sin n\xi\right)\mathrm{d}t\right\}\right], \quad 0 < \xi \leqslant \xi_0 \tag{31}$$

4.1. *Stress intensity factor*

Apart from the determination of the distribution of stress near the crack tip, stress intensity factor is also an important physical quantity in view of the fracture criteria. The stress intensity factor (SIF) is defined by the relation

$$K = \lim_{x \to x_0} (2\pi(x - x_0))^{1/2} \sigma_{yz}|_{y=0}. \tag{32}$$

By isolating the singularity and discarding the terms which are bounded as $\xi \to \xi_0$ we can easily obtain from (31)

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\frac{\pi\mu_0}{L} g(\xi_0) \frac{\sin\xi \cos\frac{1}{2}\xi}{\sin\xi_0 (\cos\xi_0 - \cos\xi)^{1/2}}\right]$$

whence from (32)

$$K = \exp(-i\Omega t)\left[\frac{\pi\mu_0}{(L\tan\frac{1}{2}\xi_0)^{1/2}} g(\xi_0)\right]. \tag{33}$$

5. Numerical results and conclusions

For numerical computation of the SIF near a crack tip we define the nondimensional time $t^* = \pi ct/L$. The SIF in nondimensional form

$$K^*(t^*) = K\sqrt{L}/\pi\mu_0 w_0 \tag{34}$$

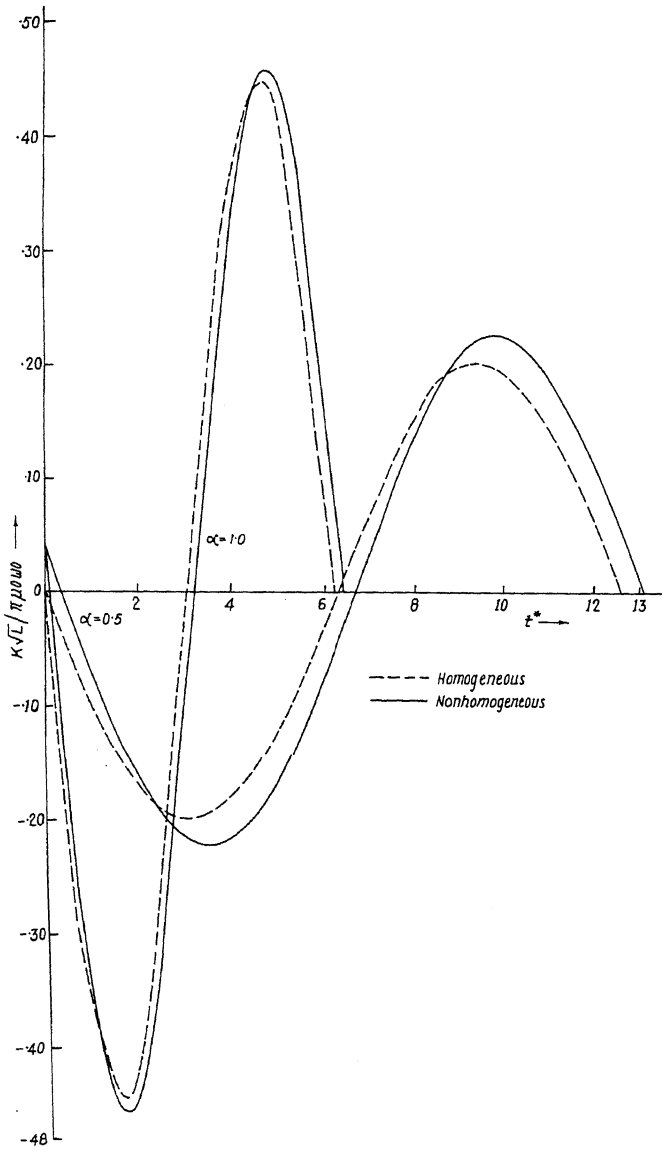

Figure 1. Stress intensity factor (K^*) as function of time (t^*) with the nonhomogeneity parameter $\delta L/2\pi = 0{\cdot}1$.

has been evaluated numerically with $\delta L/2\pi = 0{\cdot}1$ and for values of $\alpha = 0{\cdot}5$ and $\alpha = 1{\cdot}0$.

The variation of K^* with t^* has been shown in figure 1. The effect of nonhomogeneity in the diffracted wave field is found to be more prominent for small values of α than for large values of α.

Acknowledgement

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Formula for primes, twinprimes, number of primes and number of twinprimes

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Abstract. Formulae for computing the nth prime, twinprime, the number of primes smaller than a given integer, and the number of twinprimes smaller than a given integer are presented. Proofs for the development are also furnished.

Keywords. Primes; twinprimes

1. Introduction

Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_n$ be the sequence of the first n primes, and,

$$Q = \prod_{i=1}^{n} p_i \tag{1}$$

This paper presents formula to compute the $(n+1)$th prime which is p_{n+1} and other allied results.

Let

$$Q_i = Q/p_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n \tag{2}$$

$1 \leqslant a_i \leqslant (p_i - 1)$

$$K = n - \sum_{1}^{n} (1/p_i) \tag{3}$$

$0 \leqslant b \leqslant [K]$, where $[K]$ denotes the integral part of K and

$$J = \sum_{1}^{n} a_i Q_i - bQ \tag{4}$$

The following four theorems are first proved.

THEOREM 1.

For n greater than 1,

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^J = X + (1/X) + f(X) + f(1/X) \tag{5}$$

where $f(X) = \sum a_m X^m$

$$1 < m \leqslant [K]Q \prod_{i=1}^{n} (1 - (1/p_i)), \ (m, Q) = 1, \text{ and}$$

$a_m = 1$ for all $m < Q$ and $(m, Q) = 1$ and $a_m = 0$ for some $m \geqslant Q$.

THEOREM 2.

For n greater than 1,

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^{J^2} = 2X + 2g(X) \tag{6}$$

where

$$g(X) = \sum a_m X^{m^2}$$

$$1 < m \leqslant [K]Q \prod_{i=1}^{n} (1 - (1/p_i)),\ (m, Q) = 1$$

and a_m as defined in Theorem 1.

THEOREM 3.

For n greater than 1,

Let $(a_1', a_2', a_3', \ldots, a_n')$ be the unique solution of $-2 \equiv \Sigma_1^n a_i Q_i \bmod Q$, with $0 \leqslant a_i \leqslant (p_i - 1)$, $(1, 2, 3, \ldots, n)$.

Then,

$$\sum_{a_1' \neq a_1 = 1}^{(p_1-1)} \cdots \sum_{a_n' \neq a_n = 1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^J = \sum_{(m(m+2), Q) = 1} X^m \tag{7}$$

where m on the right side performs summation over the relevant range.

THEOREM 4.

For n greater than 1,

$$\sum_{a_1} \cdots \sum_{a_n} \sum_{b} X^{J^2} = \sum_{(m(m+2), Q) = 1} X^{m^2} \tag{8}$$

where m performs summation over the same set of integers as in Theorem 3.

Remark 1. Note that $a_1' = 0$ and that $1 \leqslant a_1' \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$.

Remark 2. In all the four theorems, the m's which satisfy

$$(p_n + 1) < m < (p_{n+1}^2 - 1) \tag{9}$$

are precisely all the primes in this interval.

2. Proofs

The proofs of the theorems 1 and 2 follow from the remarks given below.

First, given any integer c there is a unique solution of

$$\sum_{i=1}^{n} a_i Q_i \equiv c \bmod Q \tag{10}$$

subject to $0 \leqslant a_i \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$. The proofs of the theorems 3 and follow from the remarks given below.

Subject to $1 \leqslant a_i \leqslant (p_i - 1)$ for all i, we have secured $(m, Q) = 1$.

If in addition, the condition that $((m+2), Q) = 1$ is to be satisfied, we should have

$$\left(\left(\sum_{1}^{n} a_i Q_i - \sum_{1}^{n} a_1' Q_i\right), Q\right) = 1. \tag{11}$$

That is to say $a_i \neq a_i'$ for each i. By a similar method, it is possible to ensure

$$\{[m(m+2)(m+6)], Q\} = 1 \tag{12}$$

and so on.

Remark 3. In Theorem 2, substituting $(-J^2)$ for (J^2), we get

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{b=1}^{(p_n-1)} \sum_{b=1}^{n} X^{-J^2} = 2(X^{-1} + X^{-p_{n+1}^2} + \ldots) \tag{13}$$

where X is any given positive number.

Therefore,

$$\tfrac{1}{2}(\text{LHS}) - X^{-1} = M = (X^{-p_{n+1}^2} + X^{-p_{n+2}^2} + \ldots.) \tag{14}$$

where LHS stands for the expression on the left side of the equation.

Multiplying both sides of the above equation by $X^{p_{n+1}^2}$, we get

$$M.X^{p_{n+1}^2} = 1 + X^{-p_{n+2}^2 + p_{n+1}^2} + \ldots = 1 + R \text{ (say)}.$$

For values of $X \geqslant 2$, it is clear that $R < 1$. Therefore,

$$1 < M.X^{p_{n+1}^2} < 2 \tag{15}$$

Taking logarithms of both sides of (15), to the base X, we get

$$0 < p_{n+1}^2 + \log_X M < 1$$

$$p_{n+1}^2 > -\log_X M > p_{n+1}^2 - 1$$

That is to say, the integral part of $(-\log_X M)$ is equal to $(p_{n+1}^2 - 1)$, and,

$$p_{n+1}^2 = 1 + \left[-\log_X\left(1/2 \sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{b=1}^{n} X^{-(\Sigma_1^n a_i Q_i - bQ)^2} - (1/X)\right)\right]. \tag{16}$$

Equation (16) is thus the formula for the $(n+1)$th prime.

P is the first twinprime in (p_n, p_n^2) if P differs from zero by the equation:

$$P^2 = Y^2\left(1 - \left[\sin^2\left(\frac{\pi}{2} - [2^{-(Y^2 - p_n^4)}]\right)\right]\right) \tag{17}$$

where Y is given by the below expression:

$$Y^2 = 1 + \left[-\log_X\left(1/2 \sum_{a_1' \neq a_1 = 1}^{(p_1-1)} \cdots \sum_{a_n' \neq a_n = 1}^{(p_n-1)} \sum_{b=1}^{n} X^{-(\Sigma_1^n a_i Q_i - bQ)^2} - (1/X)\right)\right]. \tag{18}$$

It could be noted that when $Y < p_n^2$

$$P = Y \tag{19}$$

and when $Y > p_n^2$

$$P = 0.$$

Therefore, the above expression gives only twinprimes.

Remark 5. For any x in (p_n, p_n^2), the number of primes not more than x is given by

$$\Pi(x) = \frac{1}{4\pi}\int_0^{2\pi} \prod_{i=1}^{n} \frac{\sin((Q-Q_i)/2)\theta).\sin((n-1)Q\theta/2).\sin([x]\theta/2)\cos(([x]+1)\theta/2)}{\sin(Q_i\theta/2).\sin(Q\theta/2)\sin(\theta/2)} \times d\theta + n. \quad (20)$$

Noting that the angles in each set by the summation are in arithmetic progression, the above equation can be proved by summing over $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$, from,

$$\Pi(x) - \Pi(p_n) = \frac{1}{4\pi}\int_0^{2\pi} \sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{b=1}^{(n-1)} \sum_{t=1}^{[x]} \cos(\Sigma_1^n a_i Q_i - bQ)\theta \cos(t\theta)\, d\theta. \quad (21)$$

Remark 6. If $\Pi_2(x)$ represents the number of twinprimes not exceeding x; $p_n < x \leqslant p_n^2$, then,

$$\Pi_2(x) - \Pi_2(p_n) = \frac{1}{4\pi}\int_0^{2\pi} \sum_{a_1 \neq a_1'} \cdots \sum_{a_n \neq a_n'} \sum_{b=1}^{(n-1)} \sum_{t=1}^{[x]} \cos(\Sigma_1^n a_i Q_i - bQ)\theta . \cos(t\theta)\, d\theta. \quad (22)$$

Since all the twinprimes up to p_n are known, $\pi_2(p_n)$ is known.

$$\Pi_2(x) = \frac{1}{4\pi}\int_0^{2\pi} \sum_{a_1 \neq a_1'} \cdots \sum_{a_n \neq a_n'} \sum_{b=1}^{(n-1)} \sum_{t=1}^{[x]} \cos(\Sigma_1^n a_i Q_i - bQ)\theta \cos(t\theta)\, d\theta + \Pi_2(p_n). \quad (23)$$

The twinprime conjecture follows immediately if the right side (23) is greater than $\Pi_2(p_n)$

3. Detailed proofs for remarks 4 and 5

Detailed proofs for remarks 4 and 5 with numerical examples, application in connection with p_{n+1}, P, $\Pi(x)$ and $\Pi_2(x)$ will be furnished in a separate paper.

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Combined free and forced convection for laminar flow in vertical triangular channel with internal heat generation

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Abstract. The problem of combined, natural and forced convection for the laminar flow in a vertical channel of equilateral triangular cross-section is discussed. Internal heat generation is assumed to be uniform. The coupled equations in velocity and temperature are solved using equitriangular transform. Expressions for pressure drop parameter and Nusselt number are obtained and their behaviour for different values of Rayleigh number and heat generation function, is studied.

Keywords. Convection; heat generation function; pressure-drop parameter; equitriangular transform.

1. Introduction

There are many practical situations in which forced and free convections are of equal magnitude and neither can be ignored. The problem of combined free and forced convection for laminar flow has been extensively studied in vertical channels with and without internal heat generation. Hallman [4] studied the vertical circular tubes with internal heat generation, while Han [5] studied it in rectangular channels. Gupta [2, 3] investigated the unsteady laminar convective flow under a pressure gradient along a vertical pipe. The combined convection in horizontal tubes has also attracted attention. Nagendra [7] studied the interaction of free and forced convections in the transition regime. Patankar *et al* [8] discussed the effects of non-uniform heating on laminar combined convection, while El-Hawary [1] experimentally investigated the effects of combined free and forced convection on the stability of flow in a horizontal tube. The combined convection has been studied experimentally in inclined circular tubes also [9].

In the present investigation, the combined natural and forced convection has been analytically studied for fully established laminar flow in a vertical channel of equilateral triangular cross-section. Internal heat generation is also considered and assumed to be uniform. The coupled equations in velocity and temperature are solved using equitriangular transform (established on an incomplete set of eigen functions in an equilateral triangular region) due to Lu and Miller [6]. Expressions for pressure drop parameter and Nusselt number are obtained and their behaviour for different values of Rayleigh number and heat generation function is studied.

2. Formulation of the problem

The flow velocity is assumed as fully established and the heat input from the boundary of the lateral triangular channel to the fluid is constant in the flow direction. The wall

temperature is uniform in the transverse plane. The temperatures of the walls and the fluid are linear along the flow direction. It is further assumed that the fluid properties are constant unless the body force is considered. Heat dissipation due to viscosity effect is assumed to be negligible. Heat sources, providing similar distributions in all transverse planes, are also present in the fluid. The fluid flow is along the positive direction of Z-axis.

Under these conditions, the equations of motion and temperature difference are

$$\nabla^2 u = 1/\mu\,[\partial p/\partial z - F_B], \tag{1}$$

and

$$k\nabla^2 T = \rho u C_p\, \partial T/\partial Z - Q, \tag{2}$$

where u, μ, ρ, k, T, C_p, F_B, p and Q represent the axial velocity, coefficient of viscosity, density, thermal conductivity of the fluid, temperature difference between the fluid and the wall, specific heat at constant pressure, body force in positive Z direction, pressure and heat generation rate respectively.

The flow channel configuration and the co-ordinate axes are shown in figure 1.

The equations of motion and temperature, in dimensionless form, reduce to

$$\nabla^2 U + R_A \theta = -L \tag{3}$$

and

$$\nabla^2 \theta - U = F, \tag{4}$$

assuming that the fluid obeys the linear law of state and no phase change of the fluid occurs. Here U, θ, R_A, F and L represent non-dimensional velocity, temperature difference between the fluid and the boundary, Rayleigh number, heat generation

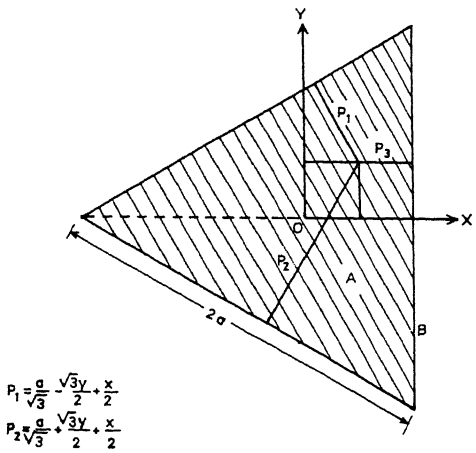

Figure 1. An equilateral triangular region.

function and pressure drop parameter respectively. The boundary conditions for the present problem are $U = 0 = \theta$, on the boundary B. (*i.e.* $p_1 = 0$, $p_2 = 0$ and $p_3 = 0$) (see figure 1).

2.1. *Method of solution*

Combining the coupled equations in velocity and temperature difference using a parameter λ, we obtain

$$\nabla^2 U + R_A\theta + \lambda(\nabla^2\theta - U - F) = -L, \tag{5}$$

or

$$\nabla^2(U + \lambda\theta) + (R_A\theta - \lambda U) = \lambda F - L. \tag{6}$$

Choosing λ such that $1/\lambda = -\lambda/R_A$ *i.e.* $\lambda = i\sqrt{R_A}$, the equation reduces to

$$(\nabla^2 + k^2)W = -L + i\sqrt{R_A}\,F, \tag{7}$$

where $W = U + i\sqrt{R_A}\,\theta$ and $k^2 = -i\sqrt{R_A}$ and the boundary conditions reduce to

$$W = 0 \text{ on the boundary B.} \tag{8}$$

Applying equitriangular transform (established on an incomplete set of eigen functions $\psi_n = \sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3$ with eigenvalues $\lambda_n = 2n\pi/\sqrt{3}a$; $n = 1, 2, 3, \ldots$) (Lu and Miller [6]), we obtain

$$-\lambda_n^2\,\tilde{W} + k^2\tilde{W} = (3\sqrt{3}a^2/n\pi)(-L + i\sqrt{R_A}F), \tag{9}$$

and

$$\tilde{W} = 0 \text{ on the boundary B,} \tag{10}$$

where $\tilde{W}$ represents equitriangular transform of W. Now, using the inversion formula for this transform, we get

$$W = \sum_{n=1}^{\infty} (2/n\pi)(\sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3)(-L + i\sqrt{R_A}F)/(k^2 - \lambda_n^2). \tag{11}$$

Equating the real and imaginary parts, we get the expressions for 2 velocity and temperature difference as

$$U = \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{L\lambda_n^2 - R_A F}{\lambda_n^4 + R_A}(\sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3) \tag{12}$$

and

$$\theta = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\frac{F\lambda_n^2 + L}{\lambda_n^4 + R_A}(\sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3). \tag{13}$$

3. Results and discussions

Figure 2 depicts the velocity distribution along the axis of x for different values of Rayleigh number (R_A) and heat generation function (F). Because of the symmetry, this distribution is the same along all the medians of a triangular section of the channel. The

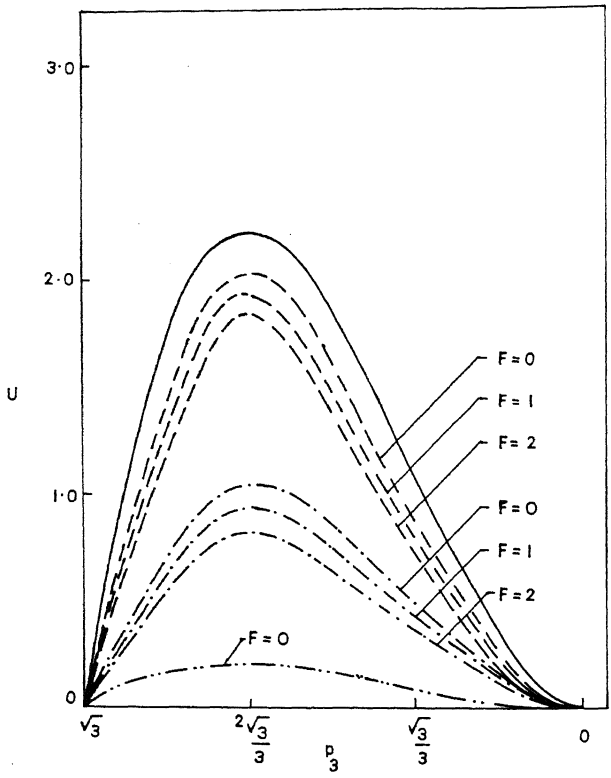

Figure 2. Velocity distribution along the axis of x for $R_A = 0$ (———), $R_A = \pi^4$ (– – –), $R_A = 10\pi^4$ (–·–·–) and $R_A = 100\pi^4$ (–·····–).

velocity of the fluid is seen to be maximum at the centroid. Increase in the Rayleigh number results in the decrease in the velocity. It is clear from the expression that for $R_A = 0$, F has no effect on the velocity. For very large values of R_A, the velocity is insignificantly affected by F. However, for the moderate values of R_A (π^4 and $10\pi^4$), the dependence on F is significant and increase in the positive values of F decreases the velocity at every point of the section.

Figure 3 shows the temperature difference between the fluid and the boundary, along the axis of x,for different values of R_A and F. It is also same along the three medians of a triangular section of the channel. For all values of R_A and F, the temperature difference is optimum at the centroid. For $F = 0$ (no heat generation in the fluid), the increase in Rayleigh number results in decrease in the temperature difference. For a particular R_A value, increase in F (heat source) gives increase in the temperature difference. A particular negative value of F (heat sink) makes the temperature of the fluid throughout the channel equal to that of the boundary. However, this negative value of F is different for different values of R_A.

4. Pressure-drop parameter

The pressure-drop parameter L is defined in terms of heat generation function F and

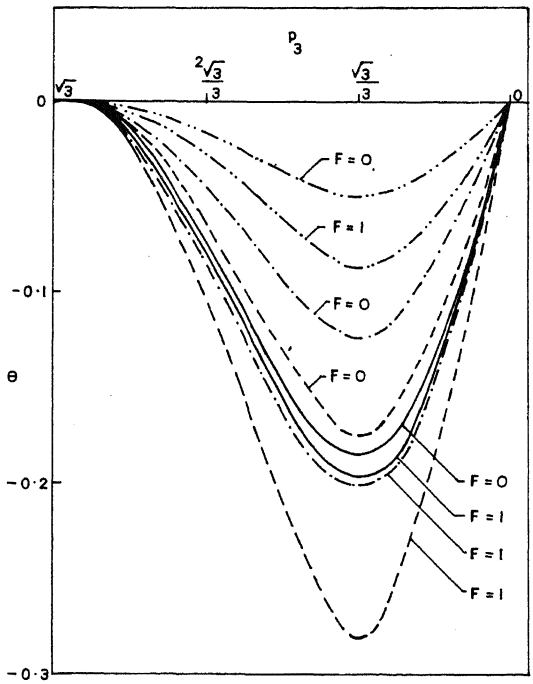

Figure 3. Temperature difference along the axis of x for $R_A = 0$ (———), $R_A = \pi^4$ (– – –), $R_A = 10\pi^4$ (–·–·–·) and $R_A = 100\pi^4$ (–··–··–··).

Rayleigh number R_A as follows:

Consider the equation of continuity,

$$\int_A U \, dx \, dy = \int_A dx \, dy, \tag{14}$$

where A is the cross-sectional area. Substituting the value of U from (12) and solving (14), we get,

$$L = \frac{1 + (6\pi^2) F R_A \sum_{n=1}^{\infty} 1/n^2 [R_A + (16n^4\pi^4/9a^4)]}{(8/a^2) \sum_{n=1}^{\infty} [1/R_A + (16n^4\pi^4/9a^4)]}, \tag{15}$$

where $2a$ is the side of the equilateral triangle.

Equation (15) shows that the pressure drop parameter is a linearly increasing function of the heat generating function F. For $a = 1$, the following expressions for the pressure drop parameter L have been obtained from (15), for different positive values of R_A.

$$\begin{aligned} L &= 20{\cdot}001 \text{ for } R_A = 0, \\ L &= 30{\cdot}072 + 6{\cdot}755\,F \text{ for } R_A = \pi^4, \\ L &= 100{\cdot}419 + 56{\cdot}354\,F \text{ for } R_A = 10\pi^4, \\ L &= 483{\cdot}215 + 363{\cdot}778\,F \text{ for } R_A = 100\,\pi^4. \end{aligned} \tag{16}$$

From (16), it is obvious that for the positive values of Rayleigh number, heat source/sink is responsible for the increase/decrease in the pressure-drop parameter. A particular negative value of F (*i.e.* heat sink) makes the pressure-drop parameter zero.

4.1. *Nusselt number*

In case of uniform heat generation, the Nusselt number is given by

$$\mathrm{Nu} = (F-1)/(-4\theta_{\mathrm{mx}}), \tag{17}$$

where θ_{mx} represents the dimensionless mixed mean temperature difference between the fluid and the wall in the same transverse plane and is expressed as

$$\theta_{\mathrm{mx}} = \int_A \theta U \,\mathrm{d}x\,\mathrm{d}y / \int_A \mathrm{d}x\,\mathrm{d}y. \tag{18}$$

Substituting the values of U and θ and solving (18), we get

$$\theta_{\mathrm{mx}} = -(2/\pi^2) \sum_{n=1}^{\infty} 1/n^2 \frac{(L+F.4n^2\pi^2/3a^2)\,(L.4n^2\pi^2/3a^2 - FR_A)}{(R_A + 16n^4\pi^4/9a^4)^2}. \tag{19}$$

For $a = 1$, the Nusselt number values are obtained for different values of heat generation function F and Rayleigh number R_A and are given in table 1.

When $F = 1$, the Nusselt number is zero and this physically means that there is no heat transfer from the walls of the channel. This represents the condition of insulated walls. Equation (13) indicates that the temperature gradient on the boundaries is not identically zero. For $F = 1$, the temperature gradient changes sign along the boundary and hence heat flows in the reverse direction also, so that the net heat transfer from the wall is zero.

From table 1, it is clear that for $F > 1$, the Nusselt number is positive *i.e.* heat flows away from the walls, while for $F < 1$, the Nusselt number is negative which means that the heat is absorbed by the fluid through the boundary.

4.2. *Rayleigh number*

In the present study, a zero Rayleigh number corresponds to either of the two cases. The first corresponds to incompressible fluid in which the free convection effects cannot occur, while the second deals with zero axial temperature gradient where free convection effects occur if the heat generation function F is non-zero. Since F and θ are infinitely large in the second case, the zero Rayleigh number corresponds to the forced convection problem only.

Table 1. Nusselt number values.

R_A	Nu for $F = 0$	$F = 0{\cdot}5$	$F = 1{\cdot}5$	$F = 2$
0	−2·333	−0·875	0·583	1·000
π^4	−2·507	−0·952	0·640	1·106
$10\pi^4$	−3·716	−1·443	1·061	1·925
$100\pi^4$	−7·171	−2·923	2·448	4·918

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An approximate solvability scheme for a class of nonlinear equations

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Abstract. An approximate solvability scheme for equations of the type $u + K_u(u) = w$, in a closed convex subset A of a Hilbert space X is given. Here, for each $u \in A$, $K_u : X \to X$ is a bounded linear operator.

Keywords. Monotone operator; Leray-Schauder principle; Chandrasekhar's H-equation.

1. Introduction

The existence of solutions of equations of the type

$$u + K_u(u) = w, \tag{1}$$

where for each $u \in A$, $K_u : X \to X$ is a bounded linear operator was earlier discussed [2]. A is a closed convex subset of a Hilbert space X. Approximate solvability of such type of equations has not yet been dealt with. Our main purpose is to give an approximation scheme for (1), which is significant from the computational point of view. As an application of our abstract result, we discuss approximate solvability of nonlinear integral equations of the form

$$u(s) + u(s) \int_0^1 K(s, t)u(t)\mathrm{d}t = w(s). \tag{2}$$

These equations include Chandrasekhar's H-equation

$$1 + u(s) \int_0^1 \frac{s}{s+t} \Psi(t)u(t)\mathrm{d}t = u(s). \tag{3}$$

Equation (3) plays an important role in the theory of radiative transfer in semi-infinite atmosphere [1], and hence the importance of our result needs no explanation.

2. Main result

Throughout this paper '$\to$' denotes strong and '$\rightharpoonup$' denotes weak convergence in a Hilbert space X. $B[O, r]$ denotes the closed ball of radius 'r' in X. For $u, w \in X$, (w, u) refers to inner product between u and w in X.

DEFINITION 2.1.

An operator $T : X \to X$ is said to be monotone if

$$(Tu - Tv, u - v) \geqslant 0 \text{ for all } u, v \text{ in } X.$$

Let $X_1 \subset X_2 \subset \ldots$ be a sequence of finite dimensional subspaces of a Hilbert space X and $P_n: X \to X_n$ a corresponding sequence of linear projections.

DEFINITION 2.2.

A tuple $\{\{X_n\}, \{P_n\}\}$ is called an approximation scheme in the space X if P_n is continuous for every n and for each $x \in X$, $P_n x \to x$ as $n \to \infty$.

In this section we shall give a constructive result concerning the existence of a solution of

$$u + K_u(u) = w \tag{4}$$

as a strong limit of solutions $u_n \in X_n$ of the 'approximate' equation

$$u_n + P_n K_{u_n}(u_n) = w_n. \tag{5}$$

DEFINITION 2.3.

Equation (4) is said to be strongly (weakly) solvable if (5) has a solution $u_n \in X_n$ and there exists a subsequence of $\{u_n\}$ which converges to u strongly (weakly) and u is a solution of (4).

Here, for $u \in X$, K_u is a bounded linear operator on X. In the following K^* denotes the conjugate of the bounded linear operator K.

THEOREM 2.1.

Let X be a Hilbert space with an approximation scheme $\{\{X_n\}, \{P_n\}\}$ and A a closed convex subset of X. Assume that for each $u \in A$, K_u is a bounded linear monotone operator satisfying the following condition:
(a) $u_k \rightharpoonup u$ in A implies that $K^*_{u_k}(v) \to K^*_u(v)$ for all $v \in X$. Then (4) is approximately strongly solvable.

Proof: We first claim that $K_u: X \to X$ is jointly weakly continuous. That is, $u_k \rightharpoonup u$ in A and $v_k \rightharpoonup v$ in X implies that $K_{u_k}(v_k) \rightharpoonup K_u(v)$. Consider $(K_{u_k}(v_k) - K_u(v), x)$, $x \in X$.

$$\begin{aligned}(K_{u_k}(v_k) - K_u(v), x) &= (K_{u_k}(v_k) - K_u(v_k), x) + (K_u(v_k) - K_u(v), x) \\ &= (v_k, K^*_{u_k}(x) - K^*_u(x)) + (K_u(v_k) - K_u(v), x).\end{aligned}$$

As $k \to \infty$ the first term in the right side of the above inequality tends to zero in view of assumption (a) and the second term tends to zero as K_u is a continuous linear operator and hence also weakly continuous. This proves our claim.

We now consider the approximate equation

$$u_n + P_n K_{u_n}(u_n) = w_n.$$

Let T be the operator on X_n defined by

$$Tu = u + P_n K_u(u) - w_n.$$

Define a closed and bounded set C as

$$C = \{u \in A \cap X_n : (w, u) \leqslant (u, u) \leqslant (w, w)\}.$$

Then T is a continuous operator on X_n such that

$$(Tu, u) = (u, u) + (P_n K_u(u), u) - (w_n, u) \geqslant (u, u) - (w, u) \geqslant 0, \text{ for } u \in C.$$

Hence it follows by Leray-Schauder principle that $Tu = 0$ has a solution $u_n \in C$. That is

$$u_n + P_n K_{u_n}(u_n) = w_n.$$

$\{u_n\}$ is a bounded sequence in a Hilbert space and hence there exists a subsequence of it, which we again denote by $\{u_n\}$, such that $u_n \rightharpoonup u$. We claim that u is a solution of (4). As $u_n \rightharpoonup u$ and $w_n \to w$ it sufficies to show that $P_n K_{u_n}(u_n) \rightharpoonup K_u(u)$ as $n \to \infty$.

Consider

$$\begin{aligned}
&(P_n K_{u_n}(u_n) - K_u(u), x),\ x \in X.\ (P_n K_{u_n}(u_n) - K_u(u), x) \\
&\quad = (P_n K_{u_n}(u_n) - P_n K_u(u), x) + (P_n K_u(u) - K_u(u), x) \\
&\quad = (K_{u_n}(u_n) - K_u(u), x) + (K_{u_n}(u_n) - K_u(u), P_n x - x) + (P_n K_u(u) \\
&\quad - K_u(u), x) \\
&\quad \leqslant (K_{u_n}(u_n) - K_u(u), x) + \|K_{u_n}(u_n) - K_u(u)\|\ \|P_n x - x\| + \|P_n K_u(u) \\
&\quad - K_u(u)\|\ \|x\|
\end{aligned}$$

$\to 0$ as $n \to \infty$, since $K_{u_n}(u_n) \rightharpoonup K_u(u)$ and $P_n x \to x$. This proves that (4) is approximately weakly solvable. We now show that $\{u_n\}$ actually converges strongly to u.

Consider

$$\begin{aligned}
\|u_n - u\|^2 &= (u_n - u, u_n - u) \\
&= (w_n - w, u_n - u) + (K_u(u) - P_n K_{u_n}(u_n), u_n - u) \\
&= (w_n - w, u_n - u) + (P_n K_{u_n}(u) - P_n K_{u_n}(u_n), u_n - u) \\
&\quad - (P_n K_{u_n}(u) - P_n K_u(u), u_n - u) - (P_n K_u(u) - K_u(u), u_n - u) \\
&= (w_n - w, u_n - u) - (K_{u_n}(u_n) - K_{u_n}(u), u_n - u) \\
&\quad - (K_{u_n}(u_n) - K_{u_n}(u), u - P_n(u)) - (K_{u_n}(u) - K_u(u), u_n - P_n u) \\
&\quad - (P_n K_u(u) - K_u(u), u_n - u) \\
&\leqslant (P_n w - w, u_n - u) - (K_{u_n}(u_n) - K_{u_n}(u), u - P_n u) \\
&\quad - (K_{u_n}(u) - K_u(u), u_n - P_n u) - (P_n K_u(u) - K_u(u), u_n - u)
\end{aligned}$$

by monotonicity of K_{u_n}.

The first, second and fourth term tend to 0 in view of $P_n x \to x$ for every $x \in X$ and the third term tend to zero in view of (a) and uniform boundedness principle. This proves our theorem.

3. Application

As an application of our main theorem we obtain an approximate solvability result for nonlinear integral equations of the type

$$u(s) + u(s) \int_0^1 K(s, t) u(t)\, dt = w(s) \tag{6}$$

in the space $L_2[0, 1]$.

We assume that $K(s, t)$ is a Hilbert-Schmidt kernel. So eigenfunctions of K form a complete orthonormal set in $L_2[0, 1]$. Let $e_1, e_2 \ldots$ be eigen functions of K with eigenvalues $\lambda_1, \lambda_2, \ldots$. Define a sequence X_n of finite dimensional subspaces of

$X = L_2[0, 1]$ and linear projections $P_n: X \to X_n$ as follows:

$$X_n = [e_1, e_2, \ldots, e_n],\ P_n u = \sum_{k=1}^{n} \alpha_k e_k \text{ where } u = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Then the approximate equation in the finite dimensional space X_n is given by

$$u(s) + P_n[u(s) \int_0^1 K(s, t)u(t)\,dt] = P_n w \tag{7}$$

where $u = \Sigma_{k=1}^{\infty} \alpha_k e_k$.

Taking inner product with e_n we get $(u, e_n) + (P_n[u(s) \int_0^1 K(s, t)\, u(t)dt], e_n) = (P_n w, e_n)$, which gives

$$\alpha_n + \left(\left(\sum_{k=1}^{n} \alpha_k e_k(s)\right)\left(\sum_{j=1}^{n} \lambda_j \alpha_j e_j(s)\right), e_n\right) = \beta_n, \tag{8}$$

where $w = \Sigma_{k=1}^{\infty} \beta_k e_k$. Writing r_{jkn} for $\int_0^1 e_k(s)e_j(s)e_n(s)\,ds$ in (8), we get an equivalent system of nonlinear equations

$$\alpha_n + \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \alpha_j \alpha_k r_{jkn} = \beta_n, \quad n \geqslant 1. \tag{9}$$

Thus solvability of the approximate equation (7) is equivalent to the solvability of (9). One can now use the known techniques to solve the nonlinear system given by (9). For various reasons we skip the details regarding the computational aspect of (9). We have the following theorem giving the approximate solvability of (6).

THEOREM 3.1

Suppose that

(a) $K(s, t) \geqslant 0$ *a.e.* on $[0, 1] \times [0, 1]$,

(b) *ess* sup $\int_0^1 K^2(s, t)dt < \infty$,

(c) $w(s) \geqslant 0$ *a.e.* on $[0, 1]$.

Then (6) is approximately strongly solvable in $L_2[0, 1]$.

Proof: Let $A = \{u \in L_2: u(s) \geqslant 0$ *a.e.* on $[0, 1]\}$ and let

$$K_u(v)(s) = v(s) \int_0^1 K(s, t)u(t)dt.$$

Then for each $u \in A$, K_u is a bounded linear operator on L_2. Also, we have $(K_u(v), v) = \int_0^1 (\int_0^1 K(s, t)u(t)dt)v^2(t)dt$

$$\geqslant 0 \quad \text{for all} \quad v \in L_2.$$

That is $\{K_u\}$ is monotone for each $u \in A$. We now verify that the hypothesis (a) of Theorem 2.1 is satisfied.

$$K_u^*(v) = v(s) \int_0^1 K(s, t)u(t)dt$$
$$= \int_0^1 K_v(s, t)u(t)dt,$$

where the new kernel $K_v(s, t) = v(s)K(s, t)$.

In view of assumption (b) of Theorem 3.1, it follows that $K_v(s, t)$ is a Hilbert-Schmidt kernel and hence the integral operator generated by it is completely continuous. That is $u_n \to u$ implies that

$$\int_0^1 K_v(s, t)u_n(t)\mathrm{d}t \to \int_0^1 K_v(s, t)u(t)\mathrm{d}t,$$

which in turn implies that $K^*_{u_n}(v) \to K^*_u(v)$. Thus the family $\{K_u\}$, $u \in A$ of linear operators on X satisfies all conditions of Theorem 2.1 and hence (6) is approximately solvable.

As a corollary of this theorem we obtain an approximiate solvability result for Chandrasekhar's H-equation

$$1 + u(s)\int_0^1 \frac{s}{s+t}\Psi(t)u(t)\mathrm{d}t = u(s). \tag{10}$$

Here, the known function $\Psi(t)$ is assumed to be non-negative, bounded and measurable. Since equation (10) is not given in the standard form we first state a lemma which is useful in this direction and for the proof refer [1].

LEMMA.

Suppose that $\int_0^1 \Psi(t)\mathrm{d}t \leqslant 1/2$ and that $u \in L_2$ is a positive solution of the equation

$$u(s)\left\{\left\{1 - 2\int_0^1 \Psi(t)\mathrm{d}t\right\}^{1/2} + u(s)\int_0^1 \frac{t}{s+t}\Psi(t)u(t)\mathrm{d}t = 1.\right. \tag{11}$$

Then $\int_0^1 \Psi(s)u(s)\mathrm{d}s = 1 - (1 - 2\int_0^1 \Psi(s)\mathrm{d}s)^{1/2}$ and u is also a solution of (10).

Equation (11) is now in the standard form (6) with

$$K(s, t) = \left[\frac{t}{c(s+t)}\right]\Psi(t) \text{ and } w(s) = \frac{1}{c}. \tag{12}$$

Here $c = [1/(1 - 2\int_0^1 \Psi(t)\mathrm{d}t)^{1/2}]$ (without loss in generality we can assume that $\int_0^1 \Psi(t)\mathrm{d}t < 1/2$, one can tackle the case $\int_0^1 \Psi(t)\mathrm{d}t = 1/2$ as a limiting case of strict inequality). Thus K and w given by (12) satisfy all the requirements of Theorem 3.1 and hence we get the following solvability result for Chandrasekhar's equation (10).

THEOREM 3.2.

Let $\int_0^1 \Psi(t)\mathrm{d}t \leqslant 1/2$, then Chandrasekhar's H-equation is approximately strongly solvable.

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Chapter 7 of Ramanujan's second notebook

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The principal topics in Chapter 7 of Ramanujan's second notebook concern sums of powers, an extended definition of Bernoulli numbers, the Riemann zeta-function $\zeta(s)$ and allied functions, Ramanujan's theory of divergent series, and the gamma function. This chapter thus represents a continuation of the subject matter of Chapters 5 and 6, which have been thoroughly examined elsewhere [6], [3]. Chapter 7 offers a considerable amount of numerical calculation. The extent of Ramanujan's calculations is amazing, since he evidently performed them without the aid of a mechanical or electrical device.

This paper provides proofs of the 110 theorems, formulas, and examples arranged in 27 sections of Chapter 7. The content of this chapter is, for the most part, correct, although there are several minor errors. We shall usually state and prove results for complex values of a variable. This is in contrast to Ramanujan who evidently intended his variables to be real. However, to give rigorous proofs, we have frequently needed to use analytic continuation, and so we state theorems in more generality than originally intended. It should be mentioned that we adhere to Ramanujan's designations of corollaries, examples, etc; these designations are frequently not optimal.

Several of the results in Chapter 7 are not new. For example, Ramanujan rediscovered the functional equation of $\zeta(s)$, found in Entry 4 in a somewhat disguised form. As was the case with Euler, Ramanujan had no real proof. It is fascinating how he arrived at this result, with reasoning based on his tenuous theory of the "constant" of a series.

Before commencing our examination of the individual results, we offer a few remarks about notation and state the Euler-Maclaurin summation formula. We shall adhere to the even suffixed notation for the Bernoulli number B_n and Euler numbers E_n, $0 \leqslant n < \infty$, as found in [1, p. 804], for example. These conventions are in contrast to those employed by Ramanujan. Thus, we define the Bernoulli numbers B_n, $0 \leqslant n < \infty$, by

$$x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k/k! \quad |x| < 2\pi. \tag{1}$$

We shall very often use the Euler-Maclaurin summation formula. If f has $2n+1$ continuous derivatives on $[\alpha, \beta]$, where α and β are integers, then [13, p. 328], [31, Chapter 13]

$$\sum_{k=\alpha}^{\beta} f(k) = \int_{\alpha}^{\beta} f(t)\,dt + \frac{1}{2}\{f(\alpha)+f(\beta)\} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\{f^{(2k-1)}(\beta) - f^{(2k-1)}(\alpha)\} + R_n, \tag{2}$$

where

$$R_n = \int_\alpha^\beta P_{2n+1}(t) f^{(2n+1)}(t)\mathrm{d}t, \qquad n \geqslant 0,$$

where P_k denotes the kth periodic Bernoulli function.

We denote complex variables by s with $\sigma = \mathrm{Re}(s)$, by r with $u = \mathrm{Re}(r)$, and by z.

ENTRY 1

Let

$$\varphi_r(x) = \sum_{k=1}^{x} k^r, \tag{3}$$

where r is any complex number. Then if $r \neq -1$, as x tends to ∞,

$$\varphi_r(x) \sim \zeta(-r) + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\Gamma(r-2k+2)}. \tag{4}$$

Proof. Applying the Euler-Maclaurin summation formula (2) with $f(t) = t^r$, $\alpha = 1$, and $\beta = x$, we find that

$$\varphi_r(x) = c + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\Gamma(r-2k+2)} - \frac{\Gamma(r+1)}{\Gamma(r-2n)} \int_x^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t,$$

where

$$c = -\frac{1}{r+1} + 1/2 - \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)}{(2k)!\Gamma(r-2k+2)} + \frac{\Gamma(r+1)}{\Gamma(r-2n)} \times \int_1^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t \tag{5}$$

and n is a positive integer with $2n > u$. Note that

$$\int_x^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t = O\left(\int_x^\infty t^{u-2n-1}\mathrm{d}t\right) = O(x^{u-2n}),$$

as $x \to \infty$. Thus, it remains to show that

$$c = \zeta(-r). \tag{6}$$

From the Euler-Maclaurin formula (2), we have, for $\sigma > 1$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(s+2k-1)}{(2k)!\Gamma(s)} - \frac{\Gamma(s+2n+1)}{\Gamma(s)} \int_1^\infty P_{2n+1}(t)t^{-s-2n-1}\mathrm{d}t. \tag{7}$$

By analytic continuation, (7) holds for $\sigma + 2n > 0$. Putting $s = -r$ in (7) and using (5), we obtain (6).

The analogue of (4) for $r = -1$ is due to Euler [13, pp. 324, 325] and is stated by Ramanujan in Chapter 8, Entry 2 [44, vol. 2, p. 91], [7].

If r is a nonnegative integer, the series in (4) is finite and we may replace the asymptotic sign by an equality sign. Moreover, (4) reduces to the familiar formula

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(!)}{r+1}, \tag{8}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial.

In Chapter 5, section 25 [44, p. 57], [6], Ramanujan defines a Bernoulli number B_r^* of arbitrary index by

$$\zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)} B_r^*. \tag{9}$$

In particular, if $r = 2n$ is an even integer, $B_{2n}^* = (-1)^{n-1} B_{2n}$ and (9) reduces to Euler's famous formula for $\zeta(2n)$. Using the functional equation for $\zeta(s)$ (see Entry 4), we find that

$$\zeta(-r) = \frac{B_{r+1}^* \cos\{\pi(r+1)/2\}}{r+1}. \tag{10}$$

In Ramanujan's version of (4), $\zeta(-r)$ is replaced by the right side of (10).

After Entry 1, Ramanujan makes some remarks about the "constant" of a series. This concept was introduced by Ramanujan in Chapter 6 and is discussed in detail in [3]. The "constant" in Entry 1 is merely the constant term $\zeta(-r)$ in the asymptotic expansion (4).

Now define, for $\sigma > 0$,

$$\eta(s) = \sum_{k=1}^{\infty} (-1)^{k+1} k^{-s}. \tag{11}$$

Note that

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \tag{12}$$

which, by analytic continuation, is valid for all complex values of s.

ENTRY 2

For each complex number r,

$$\eta(-r) = \frac{(2^{r+1} - 1)B_{r+1}^* \sin(\pi r/2)}{r+1}.$$

Proof. Set $s = -r$ in (12) and use (10).

We now wish to extend the definition of $\phi_r(x)$ to encompass complex values of x. First, for $u < 0$, redefine

$$\varphi_r(x) = \sum_{k=1}^{\infty} \{k^r - (k+x)^r\}. \tag{13}$$

Observe that, if $u < -1$,

$$\varphi_r(x) = \zeta(-r) - \psi(-r, x+1), \tag{14}$$

where $\psi(-r, x+1) = \sum_{k=1}^{\infty} (k+x)^r$. Note that $\psi(s, x)$ is very closely related to the Hurwitz zeta-function $\zeta(s, x)$, except that the latter function is usually defined only for $0 < x \leqslant 1$. The methods for analytically continuing $\zeta(s, x)$ (see, e.g., [2], [48, p. 37], [50, p. 268]) normally can be easily adapted to establish the analytic continuation of $\psi(s, x)$ as well. Thus, by analytic continuation, we shall now define $\varphi_r(x)$, for all complex values of x and r, by (14). Moreover, if $u < 0$ and x is a positive integer, we find from (13) that

$$\varphi_r(x) = \sum_{k=1}^{x} k^r. \tag{15}$$

By analytic continuation, (15) is valid for all complex values of r. Thus, the new definition (14) agrees with our former definition (3) if x is a positive integer. If r is a nonnegative integer and $-1 < x \leqslant 0$, then by (14) and the well-known fact $\psi(-r, x+1) = -B_{r+1}(x+1)/(r+1)$, we find that

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(1)}{r+1}.$$

By analytic continuation, this holds for all x, and so we see that (14) is in agreement with (8) if r is a nonnegative integer and x is complex.

COROLLARY

If r is complex and $r \neq -1$, then

$$\varphi_r(-1/2) = \frac{(2-2^{-r})B^*_{r+1}\cos\{\pi(r+1)/2\}}{r+1}.$$

Proof. By (13), if $u < 0$, we easily find that

$$\varphi_r(-1/2) = -2^{-r}\eta(-r), \tag{16}$$

where $\eta(s)$ is defined by (11). By analytic continuation, (16) is valid for all complex r, $r \neq -1$. Using Entry 2 in (16), we complete the proof.

If r is a positive integer, then, by (8), the Corollary is equivalent to the well-known fact $B_{r+1}(1/2) = -(1-2^{-r})B_{r+1}$ [1, p. 805].

ENTRY 3

Let $\varphi_r(x)$ be defined by (14) and let a and b be complex numbers with $b \neq 0$. Then

$$\sum_{k=1}^{x} (a+kb)^r = b^r\{\varphi_r(x+a/b) - \varphi_r(a/b)\}.$$

Proof. For $r < 0$, the desired formula follows easily from (13). The result for all r follows by analytic continuation.

ENTRY 4

For any complex number r.

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r} = \zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)}B^*_r. \tag{17}$$

Proof. We present Ramanujan's interesting argument, which is not rigorous.

Rewriting (8) in Ramanujan's notation, we have

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - \sin(\pi r/2)B^*_{r+1}}{r+1},$$

where r is a natural number. We now suppose that this formula is valid for *all* r. The "constant" in this representation for $\varphi_{-r}(x)$ is

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r}.$$

On the other hand, from (6) and (9), the "constant" is also equal to

$$\zeta(r) = \frac{(2\pi)^r B^*_r}{2\Gamma(r+1)}.$$

These constants must be equal, and hence (17) follows.

The equalities in (17) imply that

$$\zeta(r) = 2(2\pi)^{r-1}\Gamma(1-r)\zeta(1-r)\sin(\pi r/2). \tag{18}$$

Mirabile dictu, Ramanujan has derived the functional equation of $\zeta(r)$ [48, p. 25] in a most unorthodox manner!

COROLLARY 1

We have $B^*_2 = 2\zeta(3)$, $B^*_4 = -4\zeta(5)$, $B^*_6 = 6\zeta(7)$, and $B^*_8 = -8\zeta(9)$.

Proof. The proposed equalities are special instances of the first equality in (17).

COROLLARY 2

$$\Gamma(1/2) = \sqrt{\pi}.$$

COROLLARY 3

For every complex number z, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.

Corollary 2 is the special case $z = 1/2$ of the well-known Corollary 3 for which we give Ramanujan's proof.

Proof. Letting $r = -z$ and $r = z+1$ in the extremal sides of (17), we find, respectively, that

$$-\frac{\sin(\pi z/2)B^*_{z+1}}{z+1} = \frac{(2\pi)^{-z}B^*_z}{2\Gamma(1-z)}$$

and

$$-\frac{\cos(\pi z/2)B^*_z}{z} = \frac{(2\pi)^{z+1}B^*_{z+1}}{2\Gamma(z+2)}.$$

Multiplying these two equalities together, using the equality $\Gamma(z+2) = (z+1)z\Gamma(z)$, and simplifying, we obtain the desired result.

COROLLARY 4

We have

$$\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)^{1/2}+(2k+2)^{1/2}} = \sum_{k=0}^{\infty} 1/(2k+1)^{3/2}.$$

Ramanujan gives essentially the following faulty proof of Corollary 4. From (11) with $s = -1/2$, (12), and (18), it follows that

$$\pi \sum_{k=0}^{\infty} (-1)^k/((2k)^{1/2}+(2k+2)^{1/2}) = \frac{\pi}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k\{(k+1)^{1/2}-\sqrt{k}\}$$

$$= \pi\sqrt{2} \sum_{k=0}^{\infty} (-1)^{k+1}\sqrt{k} = \pi\sqrt{2}(1-2^{3/2})\zeta(-1/2)$$

$$= (1-2^{-3/2})\zeta(3/2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}}.$$

Corollary 4, in fact, is the special case $p = 1/2$ of the identity

$$\pi^{p+1} \sum_{k=1}^{\infty} (-1)^k\{(k+1)^p - k^p\} = 4 \sin(\pi p/2)\Gamma(p+1) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{p+1}},$$

proved in [47, p. 154], where $0 < p < 1$.

Ramanujan [41], [43, pp. 47–49] has established some other results which are akin to Corollary 4. Kesava Menon [34] has given simpler proofs of Ramanujan's results and has proved additional results of this type as well.

COROLLARY 5

Let $\eta(s)$ be defined by (11). Then

$$(2\pi)^{2/3}\eta(1/3) = (1+2^{1/3})\Gamma(2/3)\eta(2/3).$$

Proof. Using (12), we rewrite the functional equation (18) for $\zeta(r)$ in terms of $\eta(r)$ to get

$$(1-2^r)\eta(r) = 2(2\pi)^{r-1}(1-2^{1-r})\sin(\pi r/2)\Gamma(1-r)\eta(1-r).$$

Putting $r = 1/3$ and simplifying, we achieve the desired result.

COROLLARY 6

As $x \to \infty$,

$$\sum_{k=1}^{x} 1/\sqrt{k} \sim 2\sqrt{x} + \zeta(1/2) + \frac{1}{2\sqrt{x}} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3\ldots(4k-3)}{2^{2k-1}} x^{-2k+1/2}. \tag{19}$$

This asymptotic formula is a special case of (4) but is different from that claimed by Ramanujan [44, vol. 2, p. 79] who asserts that, as $x \to \infty$,

$$\sum_{k=1}^{x} 1/\sqrt{k} \sim (2+4x)^{1/2} + \zeta(1/2). \tag{20}$$

Formulas (19) and (20) are incompatible since

$$(2+4x)^{1/2} = 2\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{16x^{3/2}} + \ldots;$$

the leading two terms above agree with (19), but the third term does not coincide with the corresponding term $-x^{-3/2}/24$ in (19). Ramanujan gives no indication as to how he arrived at the approximation $(2+4x)^{1/2}$.

COROLLARY 7

As $x \to \infty$,

$$\sum_{k=1}^{x} \sqrt{k} \sim \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2} - \frac{1}{4\pi}\zeta(3/2) + \frac{1}{24}x^{-1/2} + \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3 \ldots (4k-5)}{2^{2k-1}} x^{-2k+3/2}. \tag{21}$$

Using the functional equation (18), we observe that Corollary 7 is a special instance of Entry 1, but (21) is not the result claimed by Ramanujan. Instead, he has proposed that

$$\sum_{k=1}^{x} \sqrt{k} \sim \frac{2}{3}[(x+1/4)(x+1/2)(x+3/4)]^{1/2} - \frac{1}{4\pi}\zeta(3/2), \tag{22}$$

as $x \to \infty$. Since the infinite series in (21) diverges while

$$\frac{2}{3}[(x+1/4)(x+1/2)(x+3/4)]^{1/2} = \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2} + \frac{1}{24}x^{-1/2} + \ldots$$

converges in a neighbourhood of $x = \infty$, (21) and (22) are certainly not compatible. However, note that the right side of (22) does provide a good approximation for the left side.

A similar type of approximation for $\varphi_{1/2}(x)$ has been obtained by Gates [22].

COROLLARY 8

As $x \to \infty$,

$$\sum_{k=1}^{x} k^{3/2} \sim \frac{2}{5}x^{5/2} + \frac{1}{2}x^{3/2} + \frac{1}{8}x^{1/2} - \frac{3}{16\pi^2}\zeta(5/2) - 3\sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3 \ldots (4k-7)}{2^{2k-1}} x^{-2k+5/2}. \tag{23}$$

Corollary 8 follows from Entry 1 and the functional equation (18). In contrast to (23), Ramanujan claims that

$$\sum_{k=1}^{x} k^{3/2} \sim \frac{2}{5}(x(x+1/4)(x+1/2)(x+3/4)(x+1) + \frac{5}{768}(x+1/2))^{1/2} - \frac{3}{16\pi^2}\zeta(5/2), \tag{24}$$

as $x \to \infty$. By the same reasoning used in conjunction with Corollary 7, (23) and (24) are incompatible. However,

$$\frac{2}{5}(x(x+1/4)(x+1/2)(x+3/4)(x+1) + \frac{5}{768}(x+1/2))^{1/2} = \frac{2}{5}x^{5/2} + \frac{1}{2}x^{3/2} + \frac{1}{8}x^{1/2} + \cdots.$$

ENTRY 5

Let a and b be complex numbers with $b \neq 0$ and a/b not a negative integer. Then if $u < -1$,

$$\sum_{k=1}^{\infty} (-1)^{k+1}(a+kb)^r = (2b)^r \left\{ \varphi_r\left(\frac{a}{2b}\right) - \varphi_r\left(\frac{a-b}{2b}\right) \right\}. \tag{25}$$

Proof. We have

$$\sum_{k=1}^{\infty} (-1)^{k+1}(a+kb)^r = \sum_{k=1}^{\infty} (a+(2k-1)b)^r - \sum_{k=1}^{\infty} (a+2kb)^r,$$

and the desired equality follows immediately from the definition (13).

There is a misprint in the notebooks [44, vol. 2, p. 79]; Ramanujan has written b^r instead of $(2b)^r$ on the right side of (25).

ENTRY 6(i)

Let x be a positive integer and assume that $n > 0$. Then

$$(x^2+x)^n = 2\sum_{k=0}^{\infty} \binom{n}{2k+1} \varphi_{2n-2k-1}(x).$$

Proof. The proof is indicated by Ramanujan. Expanding by the binomial theorem, we have, for $|z| \geqslant 1$ and $n > 0$,

$$\begin{aligned}(z^2+z)^n - (z^2-z)^n &= z^{2n}\{(1+1/z)^n - (1-1/z)^n\} \\ &= z^{2n}\left\{ \sum_{k=0}^{\infty} \binom{n}{k} z^{-k} - \sum_{k=0}^{\infty} \binom{n}{k} (-z)^{-k} \right\} \\ &= 2z^{2n} \sum_{k=0}^{\infty} \binom{n}{2k+1} z^{-2k-1}. \end{aligned} \tag{26}$$

Now set $z = j$ in (26) and sum both sides on j, $1 \leqslant j \leqslant x$, to get

$$(x^2+x)^n = 2\sum_{j=1}^{x} j^{2n} \sum_{k=0}^{\infty} \binom{n}{2k+1} j^{-2k-1}.$$

The required formula follows upon inverting the order of summation and employing equation (3).

ENTRY 6(ii)

Under the same hypotheses as Entry 6(i), we have

$$(x+1/2)(x^2+x)^n = \sum_{k=0}^{\infty} \left\{ 2\binom{n}{2k+1} + \binom{n}{2k} \right\} \varphi_{2n-2k}(x).$$

Proof. Proceeding in the same fashion as in the previous proof, we find that, for $|z| \geqslant 1$ and $n > 0$,

$$(z+1/2)(z^2+z)^n - (z-1/2)(z^2-z)^n$$
$$= z^{2n} \sum_{k=0}^{\infty} \left\{ 2\binom{n}{2k+1} + \binom{n}{2k} \right\} z^{-2k}. \tag{27}$$

Letting $z = j$ in (27) and summing both sides on j, $1 \leqslant j \leqslant x$, we arrive at the formula that we sought with no difficulty.

Note that if n is a positive integer, then Entries 6(i) and 6(ii) are valid for all x because they yield polynomial identities.

COROLLARY 1

Let $y = x^2 + x$ and $a = x + 1/2$. Then

$$\varphi_1(x) = \tfrac{1}{2}y, \qquad \varphi_2(x) = \tfrac{1}{3}ay, \qquad \varphi_3(x) = \tfrac{1}{4}y^2,$$
$$\varphi_4(x) = \tfrac{1}{5}ay(y-\tfrac{1}{3}), \qquad \varphi_5(x) = \tfrac{1}{6}y^2(y-\tfrac{1}{2}),$$
$$\varphi_6(x) = \tfrac{1}{7}ay(y^2-y+\tfrac{1}{3}), \qquad \varphi_7(x) = \tfrac{1}{8}y^2(y^2-\tfrac{4}{3}y+\tfrac{2}{3}),$$
$$\varphi_8(x) = \tfrac{1}{9}ay(y^3-2y^2+\tfrac{9}{5}y-\tfrac{3}{5}),$$
$$\varphi_9(x) = \tfrac{1}{10}y^2(y-1)(y^2-\tfrac{3}{2}y+\tfrac{3}{2}),$$
$$\varphi_{10}(x) = \tfrac{1}{11}ay(y-1)(y^3-\tfrac{7}{3}y^2+\tfrac{10}{3}y-\tfrac{5}{3}),$$

and

$$\varphi_{11}(x) = \tfrac{1}{12}y^2(y^4-4y^3+\tfrac{17}{2}y^2-10y+5).$$

Proof. The proposed odd indexed formulas for $\varphi_r(x)$ follow from Entry 6(i) by successively letting $n = 1, 2, \ldots, 6$. The proposed even indexed formulas arise from Entry 6(ii) by successively setting $n = 1, 2, \ldots, 5$.

Although the formulas in Corollary 1 have long been known and are instances of (8), Ramanujan's method for determining them by means of Entries 6(i) and 6(ii) is particularly brief and elegant. For other formulas and methods for finding $\varphi_r(x)$ when r and x are positive integers, see a survey paper by Snow [45] which contains several references.

COROLLARY 2

For each positive integer n, we have

(i) $$\sum_{k=1}^{n} \left(\frac{2k-1+\sqrt{5}}{9} \right)^9 = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2} \right)$$

and

(ii) $$\sum_{k=1}^{n} \left(\frac{2k-1+\sqrt{5}}{2} \right)^{10} = \varphi_{10}\left(\frac{2n-1+\sqrt{5}}{2} \right).$$

If p and n are positive integers with n even, then

(iii) $$\sum_{k=1}^{p} (2k-1)^n = 2^n \varphi_n(p-1/2).$$

Proof. Applying Entry 3 with $r = 9, x = n, a = (\sqrt{5}-1)/2$, and $b = 1$, we find that

$$\sum_{k=1}^{n}\left(\frac{2k-1+\sqrt{5}}{2}\right)^9 = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2}\right) - \varphi_9\left(\frac{\sqrt{5}-1}{2}\right).$$

However, by Corollary 1, it is easily seen that $(\sqrt{5}-1)/2$ is a root of $\varphi_9(x)$. Hence, part (i) follows.

Part (ii) follows in the same fashion as part (i), except now we use the fact that $\varphi_{10}((\sqrt{5}-1)/2) = 0$.

To prove (iii), apply Entry 3 with $r = n$, $x = p$, $a = -1$, and $b = 2$ to get

$$\sum_{k=1}^{p}(2k-1)^n = 2^n\{\varphi_n(p-1/2) - \varphi_n(-1/2)\}.$$

By the Corollary to Entry 2, $\varphi_n(-\frac{1}{2}) = 0$, and the proof is complete.

ENTRY 7

If r is a positive integer, then

$$\varphi_r(x-1) + (-1)^r\varphi_r(-x) = 0.$$

Proof. By (8),

$$\varphi_r(x-1) + (-1)^r\varphi_r(-x) = \frac{B_{r+1}(x) + (-1)^r B_{r+1}(1-x)}{r+1}.$$

By a very familiar property of Bernoulli polynomials [1, p. 804], the right side above is equal to 0.

COROLLARY

If r is a positive integer exceeding 1, then $\varphi_r(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as r is odd or even.

Proof. This result follows easily from Entries 6(i), (ii) by induction on r.

ENTRY 8

If r is a positive integer, then

$$\begin{aligned}\varphi_r(x) &= \frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k}B_k x^{r+1-k} + x^r \\ &= \frac{x^{r+1}}{r+1} + \frac{x^r}{2} - 2\sum_{k=1}^{[r/2]}\binom{r+1}{2k}k\zeta(1-2k)x^{r+1-2k}. \qquad (28)\end{aligned}$$

Proof. Using the well-known formula [1, p. 804]

$$B_r(x) = \sum_{k=0}^{r}\binom{r}{k}B_k x^{r-k}, r \geqslant 0,$$

in (8), we find that

$$\varphi_r(x) = \varphi_r(x-1) + x^r = \frac{1}{r+1}\sum_{k=0}^{r+1}\binom{r+1}{k}B_k x^{r+1-k} - \frac{B_{r+1}}{r+1} + x^r,$$

from which the first equality of (28) readily follows. The latter equality of (28) follows from (17).

ENTRY 9. is simply a restatement of (13).

ENTRY 10.

For each complex number r and each positive integer n,

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right) = (1-n^{r+1})\zeta(-r)$$

$$= (n^{r+1}-1)\frac{\sin(\pi r/2)B^*_{r+1}}{r+1}. \tag{29}$$

Proof. For $u < -1$, (13) yields

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right)$$

$$= \zeta(-r) - \sum_{j=1}^{\infty} (j+x)^r - n^{r+1}\zeta(-r) + \sum_{k=0}^{n-1}\sum_{j=1}^{\infty} (nj-k+x)^r$$

$$= (1-n^{r+1})\zeta(-r). \tag{30}$$

By analytic continuation, the extremal sides of (30) are equal for all r. The second equality in (29) follows from (17).

COROLLARY

Under the hypotheses of Entry 10,

$$\sum_{k=1}^{n-1} \varphi_r(-k/n) = (n-n^{-r})\zeta(-r).$$

Proof. Put $x = 0$ in Entry 10 and use the fact that $\varphi_r(0) = 0$ for each r.

ENTRY 11

If r is a positive integer, then

$$\varphi_{-r}(x-1) + (-1)^r \varphi_{-r}(-x)$$

$$= \{1+(-1)^r\}\zeta(r) + \frac{(-1)^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}(\pi\cot(\pi x)), \tag{31}$$

where if $r = 1$, the first expression on the right side of (31) is understood to be equal to 0.

Proof. By (13),

$$\varphi_{-r}(x-1) + (-1)^r \varphi_{-r}(-x)$$

$$= \{1+(-1)^r\}\zeta(r) - \sum_{k=0}^{\infty}\left\{\frac{1}{(k+x)^r} + \frac{1}{(x-k-1)^r}\right\}.$$

Since

$$\pi \cot(\pi x) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k+x} + \frac{1}{x-k-1} \right\},$$

equality (31) now easily follows.

In the notebooks [44, vol. 2, p. 81], Ramanujan gives (31) with r replaced by $-r$ and states that the result is obtained by differentiating the equality $\varphi_{-1}(x-1) - \varphi_{-1}(-x) = -\pi \cot(\pi x)$ r times. The correct number of differentiations is however $-r-l$. Ramanujan then indicates that (31) holds for negative as well as positive values of r and that Entry 7 can thus be deduced. If we interpret

$$\frac{(-1)^r}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} \cot(\pi x)$$

as being identically 0 for $r < 0$, then, indeed, we obtain Entry 7 (but this does *not prove* Entry 7).

Ramanujan next indicates a method for calculating the derivatives of $\cot(\pi x)$. We are not certain what Ramanujan's method is, but it seems to be a more complicated version of the simple method which we describe below. This method has been judiciously applied and generalized by Carlitz and Scoville [14], [15]. Set $y = \cot(\pi x)$. Then $\tan^{-1}(1/y) = \pi x$. Upon differentiating both sides of the latter equality with respect to x, we find that

$$\frac{dy}{dx} = -\pi(y^2+1). \tag{32}$$

Further derivatives of $\cot(\pi x)$ can be found by successively differentiating (32). In this manner, the following table of derivatives of $\cot(\pi x)$ may rapidly be calculated. All formulas are correctly given by Ramanujan, except that he has written 2385 for 2835 in the last denominator of the last entry.

k	$\frac{(-1)^k}{\pi^k k!} \frac{d^k y}{dx^k}$
0	y
1	y^2+1
2	y^3+y
3	$y^4+\frac{4}{3}y^2+\frac{1}{3}$
4	$y^5+\frac{5}{3}y^3+\frac{2}{3}y$
5	$y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15}$
6	$y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y$
7	$y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315}$
8	$y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y$
9	$y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835}$

COROLLARY

If r is any complex number, then

(i) $$\varphi_r(x)-2^r\left\{\varphi_r\left(\frac{x}{2}\right)+\varphi_r\left(\frac{x-1}{2}\right)\right\}=(1-2^{r+1})\zeta(-r),$$

(ii) $$\varphi_r(-1/2)=(2-2^{-r})\zeta(-r),$$

(iii) $$\varphi_r(-1/3)+\varphi_r(-2/3)=(3-3^{-r})\zeta(-r),$$

(iv) $$\varphi_r(-1/4)+\varphi_r(-3/4)=(2+2^{-r}-4^{-r})\zeta(-r),$$

and

(v) $$\varphi_r(-1/6)+\varphi_r(-5/6)=(1+2^{-r}+3^{-r}-6^{-r})\zeta(-r).$$

Proof. Part (i) is the case $n=2$ of Entry 10. Parts (ii)–(v) follow from the Corollary to Entry 10 by successively setting $n=2$, 3, 4 and 6, respectively.

EXAMPLES

If r is a positive, odd integer, then

(i) $$\varphi_r(-1/3)=(3-3^{-r})\zeta(-r)/2,$$

(ii) $$\varphi_r(-1/4)=(1+2^{-r-1}-2^{-2r-1})\zeta(-r),$$

(iii) $$\varphi_r(-1/6)=(1+2^{-r}+3^{-r}-6^{-r})\zeta(-r)/2,$$

(iv) $$\varphi_r(-1/5)+\varphi_r(-2/5)=(5-5^{-r})\zeta(-r)/2,$$

(v) $$\varphi_r(-1/8)+\varphi_r(-3/8)=(2+2^{-2r-1}-2^{-3r-1})\zeta(-r),$$

(vi) $$\varphi_r(-1/10)+\varphi_r(-3/10)=(3+2^{-r}+5^{-r}-10^{-r})\zeta(-r)/2,$$

and

(vii) $$\varphi_r(-1/12)+\varphi_r(-5/12)=(4-2^{-r}+4^{-r}+6^{-r}-12^{-r})\zeta(-r)/2.$$

Proof. All of these formulas are easily established with the use of the Corollary of Entry 10 and Entry 7. For illustration, we shall give the proof of part (vii). By the aforementioned results,

$$(12-12^{-r})\zeta(-r)=\sum_{k=1}^{11}\varphi_r(-k/12)=2\sum_{k=1}^{5}\varphi_r(-k/12)+\varphi_r(-1/2).$$

Using Examples (i), (ii) and (iii) and Corollary (ii), we find that

$$\begin{aligned}\varphi_r(-1/12)+\varphi_r(-5/12)&=(12-12^{-r})\zeta(-r)/2\\&\quad-(3-3^{-r})\zeta(-r)/2-(1+2^{-r-1}-2^{-2r-1})\zeta(-r)\\&\quad-(1+2^{-r}+3^{-r}-6^{-r})\zeta(-r)/2-(2-2^{-r})\zeta(-r)/2,\end{aligned}$$

which, upon simplification, yields (vii).

Ramanujan has incorrectly given the right sides of (vi) and (vii) [44, vol. 2, p. 83]. The examples above are more commonly expressed in terms of values of Bernoulli polynomials. For example, see [1, pp. 805, 806].

ENTRY 12

For every complex number r,

$$2^r\{\varphi_r(-1/6)-\varphi_r(-5/6)\} = (2^r+1)\{\varphi_r(-1/3)-\varphi_r(-2/3)\}. \tag{33}$$

Proof. Putting $n=2$ and $x=-1/3$ and $x=-2/3$ in Entry 10, we find that, respectively,

$$\varphi_r(-1/3)-2^r\{\varphi_r(-1/6)+\varphi_r(-2/3)\} = (1-2^{r+1})\zeta(-r) \tag{34}$$

and

$$\varphi_r(-2/3)-2^r\{\varphi_r(-1/3)+\varphi_r(-5/6)\} = (1-2^{r+1})\zeta(-\mathrm{r}). \tag{35}$$

Subtracting (35) from (34) and rearranging terms, we deduce (33).

The proof above is given by Ramanujan in the notebooks, but he has inadvertently multiplied the right sides of (34) and (35) by -1.

EXAMPLE 1

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3).$$

EXAMPLE 2

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)^3} = \frac{2\pi^3}{81\sqrt{3}}+\frac{13}{27}\zeta(3).$$

EXAMPLE 3

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)^3} = \frac{\pi^3}{64}+\frac{7}{16}\zeta(3).$$

EXAMPLE 4

$$\sum_{k=0}^{\infty}\frac{1}{(6k+1)^3} = \frac{\pi^3}{36\sqrt{3}}+\frac{91}{216}\zeta(3).$$

All of these examples follow from well-known general formulas. Example 1 is trivial. Examples 2–4 follow from general formulas for $\varphi_{-2n-1}(-2/3)$, $\varphi_{-2n-1}(-3/4)$, and $\varphi_{-2n-1}(-5/6)$ that can be found in Hansen's tables [28, formulas (6.3.10), (6.3.18), and (6.3.23), pp. 118, 119].

ENTRY 13

For each nonnegative integer k, define

$$c_k = \lim_{m\to\infty}\left(\sum_{j=1}^{m}\frac{\log^k j}{j}-\frac{\log^{k+1}m}{k+1}\right). \tag{36}$$

Then for all s,

$$\zeta(s) = \frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k c_k}{k!}(s-1)^k. \tag{37}$$

In particular, if $A_k = (-1)^k c_k/k!$, $0 \leqslant k < \infty$, then

$$A_0 = \gamma = 0{\cdot}5772156649,$$

$$A_1 = 0{\cdot}0728158455, \quad A_2 = -0{\cdot}00485, \quad \text{and} \quad A_3 = -0{\cdot}00034, \tag{38}$$

where γ denotes Euler's constant.

Ramanujan did not explicitly define c_k by (36). Instead, he says that c_k is the constant of $\Sigma_{j=1}^{\infty} (\log^k j/j)$, but this is equivalent to (36). The values of A_k, $0 \leqslant k \leqslant 3$, are correct to the given number of decimal places.

The Laurent series (37) has been independently discovered several times in the literature. Apparently, Stieltjes [46] first established (37) in 1885. Furthermore, Stieltjes and Hermite [46, letters 73, 74, 75, 77] have thoroughly discussed this result in an exchange of letters. Not surprisingly, the constants A_k are now called Stieltjes constants. In 1887, Jensen [33] rediscovered (37). Hardy [30], [32, pp. 475–476] and Ramanujan [42], [43, p. 134] himself each stated (37) without proof. Briggs and Chowla [12] rediscovered (37) again in 1955. Later proofs have been given by Verma [49] and Ferguson [19] in 1963 and Lammel [38] in 1966. Kluyver [36] has established an infinite series representation for c_k.

Wilton [51] and Berndt [2] have evaluated the Laurent coefficients of the Hurwitz zeta-function. Further generalizations to other Dirichlet series have been found by Briggs and Buschman [11] and Knopfmacher [37].

Numerical calculations of the constants c_k were first carried out by Jensen [33] who calculated the first 9 coefficients to 9 decimal places. In 1895, Gram [26] published a table of the first 16 coefficients to 16 decimal places. The most extensive calculations todate have been by Liang and Todd [39] who calculated the first 20 coefficients to 15 decimal places.

Briggs [10] and Mitrović [40] have proved theorems on the signs of the coefficients c_k. Uniform bounds for $|c_k|$ have been established by Briggs [10]; the best estimates to date are due to Berndt [2].

EXAMPLE 1

For $|n|$ sufficiently small, we have

$$\zeta(1+n) + \zeta(1-n) = \frac{2\gamma}{1 + 0{\cdot}00839\, n^2 - 0{\cdot}0001\, n^4 + \ldots}.$$

Proof. From Entry 13, for $|n|$ sufficiently small,

$$\zeta(1+n) + \zeta(1-n) = 2\gamma + 2A_2 n^2 + 2A_4 n^4 + \ldots$$

$$= \frac{2\gamma}{1 - \dfrac{A_2}{\gamma} n^2 + \left(\dfrac{A_2^2}{\gamma^2} - \dfrac{A_4}{\gamma} \right) n^4 + \ldots}.$$

Using the values of A_0, A_2 and A_4 given by Liang and Todd [39] and employing a calculator, we complete the proof.

In Example 1 Ramanujan, in fact, has written $+0{\cdot}0001\, n^4$ instead of $-0{\cdot}0001\, n^4$. Several of the following examples also need corrections.

EXAMPLE 2

$$\zeta(11/10) = 10{\cdot}58444842.$$

EXAMPLE 3

$$\zeta(3/2) = 2{\cdot}6123752.$$

EXAMPLE 4

$$\zeta(5/2) = 1{\cdot}341490.$$

EXAMPLE 5

$$B^*_{3/2} = 0{\cdot}4409932;\ B^*_{1/2} = -1{\cdot}032627.$$

EXAMPLE 6

$$B^*_{1/3} = -0{\cdot}9420745;\ B^*_{-1/3} = -1{\cdot}3841347.$$

EXAMPLE 7

$$B^*_{-1/2} = -1{\cdot}847228.$$

According to Gram's [27] table of values for $\zeta(s)$ which has been reproduced in Dwight's tables [18], the last recorded digit for $\zeta(11/10)$ should be 6 rather than 2. These same tables indicate that the last recorded digit for $\zeta(3/2)$ is 3 and not 2 and that the last two digits of $\zeta(5/2)$ are 87 instead of 90. In an earlier table of Glaisher [23], the values of $\zeta(11/10)$ and $\zeta(3/2)$ are found to six decimal places.

The five particular values of B^*_r given by Ramanujan can be found by employing (17) in conjunction with tabulated values of the Riemann zeta-function. Using the value of $\zeta(3/2)$, we find that the last digit of $B^*_{3/2}$ should be 3 rather than 2. The given values of $B^*_{1/2}$ and $B^*_{-1/2}$ are correct. To calculate $B^*_{1/3}$ and $B^*_{-1/3}$ we need the values of $\zeta(2/3)$ and $\zeta(4/3)$ which are not found in the aforementioned tables but which have been calculated by Hansen and Patrick [29]. Accordingly, the last digit of $B^*_{1/3}$ should be 3 rather than 5. Ramanujan's value of $B^*_{-1/3}$, in contrast to his other calculations, is somewhat off from the correct value $-1{\cdot}3860016$.

For a list of all tables of the Riemann zeta-function before 1962, consult the Index of Fletcher *et al* [20]. The most extensive computations of $\zeta(s)$ appear to have been done by McLellan in 1968; see Wrench's review [52] for a description of these tables.

ENTRY 14

Let $n > 0$. Then as $n \to 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k(k^n-1)} \sim \frac{a_0 - \log n}{n} + a_1 + \sum_{k=1}^{\infty} a_{k+1} n^{2k-1}, \tag{39}$$

where

$$a_0 = \lim_{m\to\infty} \left(\sum_{k=2}^{m} \frac{1}{k \log k} - \log\log m \right) = 0{\cdot}7946786, \tag{40}$$

$$a_1 = \tfrac{1}{2}(1-\gamma) = 0{\cdot}2113922,$$

and

$$a_{k+1} = \frac{-B_{2k}A_{2k-1}}{2k}, \quad k \geqslant 1,$$

where B_j denotes the jth Bernoulli number and A_j is defined in Entry 13, $0 \leqslant j < \infty$. In particular, $a_2 = -0{\cdot}0060680$ and $a_3 = -0{\cdot}000000475$.

The numerical value for $a_0 + \log\log 2$ is found in an article of Boas [9, p. 156]. Boas records the first six digits of $a_0 + \log\log 2$ in [8, p. 244]. The numerical values for a_1, a_2, and a_3 may be determined from (38), or, more accurately, from the table of Liang and Todd [39]. While Ramanujan correctly gives a_1 and a_2, his value $-0{\cdot}0000028$ for a_3 is incorrect.

Proof. Let $t \geqslant 1$, $x \geqslant 0$, and suppose that $0 < n < A$, where A is fixed and positive but otherwise arbitrary. Define

$$f(t) = \frac{1}{t(t^n - 1)}, \quad h(t) = f(t) - \frac{1}{nt\log t}, \text{ and } g(x) = \frac{1}{e^x - 1} - \frac{1}{x}.$$

Then $h(t) = g(n\log t)/t$ and

$$h'(t) = t^{-2}\{ng'(n\log t) - g(n\log t)\}. \tag{41}$$

Fix an integer $N \geqslant 1$. Applying Taylor's theorem to g and g', we see from (41) that

$$t^2h'(t) = \sum_{j=0}^{N} \frac{(n\log t)^j}{j!}\{ng^{(j+1)}(0) - g^{(j)}(0)\} + \frac{(n\log t)^{N+1}}{(N+1)!}\{ng^{(N+2)}(\theta_1) - g^{(N+1)}(\theta_2)\}, \tag{42}$$

where $0 \leqslant \theta_1, \theta_2 \leqslant n\log t$. By the definition (1) for the Bernoulli numbers,

$$g^{(j)}(0)/j! = B_{j+1}/(j+1)!, \quad j \geqslant 0. \tag{43}$$

Thus, as $x \to 0$, $g^{(N)}(x) \to B_{N+1}/(N+1)$. Using the fact that

$$g(x) = -\frac{1}{x} + \sum_{k=0}^{\infty} \exp(-(k+1)x), \quad x > 0,$$

we find that $g^{(N)}(x) \to 0$ as $x \to \infty$. Hence, $g^{(N)}(x)$ is bounded for each fixed N, and so the last expression in (42) is $0(\{n\log t\}^{N+1})$, where the implied constant is independent of n and t. Using (43), we deduce from (42) that

$$t^2h'(t) = \sum_{j=0}^{N} \frac{g^{(j)}(0)}{j!}\{nj(n\log t)^{j-1} - (n\log t)^j\} + 0(\{n\log t\}^{N+1})$$

$$= t^2 \sum_{j=0}^{N} \frac{B_{j+1}n^j}{(j+1)!}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\log^j t}{t}\right) + 0(\{n\log t\}^{N+1}). \tag{44}$$

Next, apply the Euler-Maclaurin formula (2) with 0, $h(t)$, 1, and m playing the roles of n, $f(t)$, α, and β, respectively. Since $h(1) = -1/2$, we find that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,\mathrm{d}t + 1/4 + \frac{h(m)}{2} + \int_1^m P_1(t)h'(t)\,\mathrm{d}t. \tag{45}$$

From (44) and (45), it follows that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,dt + 1/4 + \frac{h(m)}{2} + \sum_{j=0}^{N} \frac{B_{j+1} n^j}{(j+1)!} \int_1^m P_1(t) \frac{d}{dt}\left(\frac{\log^j t}{t}\right) dt + 0(n^{N+1}), \tag{46}$$

where the implied constant is independent of n and m. We now evaluate the integrals on the right side of (46). First,

$$\int_1^m h(t)\,dt = \frac{1}{n}\log\left(\frac{t^n-1}{t^n \log t}\right)\Bigg|_1^m = \frac{1}{n}\log\left(\frac{m^n-1}{m^n}\right) - \frac{\log\log m}{n} - \frac{\log n}{n}. \tag{47}$$

By the Euler-Maclaurin formula (2),

$$\int_1^m P_1(t)\frac{d}{dt}\left(\frac{\log^j t}{t}\right)dt = \begin{cases} \displaystyle\sum_{k=1}^{m} \frac{\log^j k}{k} - \int_1^m \frac{\log^j t}{t}\,dt - \frac{\log^j m}{2m} = \sum_{k=1}^{m} \frac{\log^j k}{k} - \frac{\log^{j+1} m}{j+1} - \frac{\log^j m}{2m}, & j > 0, \\ \displaystyle\sum_{k=1}^{m} \frac{1}{k} - \int_1^m \frac{dt}{t} - \frac{1}{2}\left(1+\frac{1}{m}\right) = \sum_{k=1}^{m} \frac{1}{k} - \log m - \frac{1}{2}\left(1+\frac{1}{m}\right), & j = 0. \end{cases} \tag{48}$$

Using the integral evaluations (47) and (48) in (46) and then letting $m \to \infty$, we find that

$$\sum_{k=2}^{\infty} f(k) = \frac{a_0}{n} - \frac{\log n}{n} + 1/4 + B_1(\gamma - 1/2) + \sum_{j=1}^{N} \frac{B_{j+1} n^j c_j}{(j+1)!} + 0(n^{N+1}),$$

where c_j is defined in (36). The asymptotic formula (39) now readily follows.

Corollary 1 is a restatement of (40).

COROLLARY 2

For $s > 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k^{s+1}\log k} = -\log s + C + (1-\gamma)s - \sum_{k=2}^{\infty} \frac{A_{k-1} s^k}{k}, \tag{49}$$

where

$$C = \sum_{k=2}^{\infty} \frac{1}{k^2 \log k} - 1 + \gamma + \sum_{k=2}^{\infty} \frac{A_{k-1}}{k}, \tag{50}$$

and where A_k, $1 \leqslant k < \infty$, is defined in Entry 13. Furthermore, $C = 0{\cdot}2174630$, $1-\gamma = 0{\cdot}4227843$, $-\frac{1}{2}A_1 = -0{\cdot}0364079$, $-\frac{1}{3}A_2 = 0{\cdot}001615$, $-\frac{1}{4}A_3 = 0{\cdot}000086$, and $-\frac{1}{5}A_4 = -0{\cdot}00002$.

Proof. Replacing s by $x+1$ in (37) and integrating over $[1, s]$, we find that, for $s > 0$,

$$-\sum_{k=2}^{\infty}\frac{1}{k^{s+1}\log k}+\sum_{k=2}^{\infty}\frac{1}{k^2\log k}+s-1$$
$$=\int_1^s\zeta(x+1)\,dx=\log s+\gamma(s-1)+\sum_{k=2}^{\infty}\frac{A_{k-1}(s^k-1)}{k}.$$

Hence, (49) and (50) follow immediately.

The numerical coefficients of s^k, $1\leqslant k\leqslant 5$, are now found by employing the table of Liang and Todd [39]. The value

$$\sum_{k=2}^{\infty}\frac{1}{k^2\log k}=0{\cdot}605521788883$$

was calculated by J R Hill on his PDP11/34 computer. Using this computation along with the table from [39] and the bounds $|A_k|\leqslant 4/(k\pi^k)$, $1\leqslant k<\infty$, found in a paper of Berndt [2], we derive the proposed value of C.

Ramanujan's version of Corollary 2 contains some minor discrepancies; his coefficients of s^3 and s^4 are 0·001617 and 0·000085, respectively.

ENTRY 15

Let $u>-1$ and $0<x<1$. Then

$$\frac{\varphi_r(x-1)-\varphi_r(-x)}{4\Gamma(r+1)}=-\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{r+1}}.$$

Proof. Recall Hurwitz's formula [48, p. 37]

$$\zeta(s,a)=2\Gamma(1-s)\left\{\sin(\pi s/2)\sum_{k=1}^{\infty}\frac{\cos(2\pi ka)}{(2\pi k)^{1-s}}+\cos(\pi s/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi ka)}{(2\pi k)^{1-s}}\right\},\tag{51}$$

where $\sigma<1$ and $0<a\leqslant 1$. By (14) and (51), the desired formula readily follows.

ENTRY 16

Let $u>-1$ and $0<x<1$. Then

$$\frac{\varphi_r(x-1)+\varphi_r(-x)-2\zeta(-r)}{4\Gamma(r+1)}=\sin(\pi r/2)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{r+1}}.$$

Proof. The proof is completely analogous to the previous proof.

COROLLARY 16(i)

Let p and q be integers with $0<p<q$. Then if r is any complex number,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$
$$=-\sin(\pi r/2)\sum_{j=1}^{q-1}\sin(2\pi jp/q)\left\{\zeta(r)-\varphi_{-r}\left(\frac{j}{q}-1\right)\right\}.\tag{52}$$

Proof. Using Entry 15 and putting $k=mq+j$, $1\leqslant j\leqslant q$, $0\leqslant m<\infty$, we find that, for $r>1$,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\} = -\sin(\pi r/2)q^r\sum_{k=1}^{\infty}\frac{\sin(2\pi kp/q)}{k^r}$$

$$= -\sin(\pi r/2)\sum_{j=1}^{q}\sin(2\pi jp/q)\sum_{m=0}^{\infty}\frac{1}{(m+j/q)^r}.$$

The result now follows from (14) for $r > 1$ and by analytic continuation for all r.

COROLLARY 16(ii)

Let p and q be integers with $0 < p < q$. For any complex number r,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2(1-q^{-r})\zeta(1-r)\right\}$$

$$= -\cos(\pi r/2)\sum_{j=1}^{q-1}\cos(2\pi jp/q)\left\{\zeta(r)-\varphi_{-r}\left(\frac{j}{q}-1\right)\right\}. \tag{53}$$

Proof. The proof is similar to the previous proof, but, in addition, uses the functional equation (18).

For the next few results we shall need Ramanujan's extended concept of the Euler numbers (see [6, section 25]). Define

$$E_r^* = \frac{2\Gamma(r)}{(\pi/2)^r}L(r) \tag{54}$$

where r is any complex number, and where

$$L(s) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-s}, \quad \sigma > 0. \tag{55}$$

It is well-known [16, Chapter 9] that $L(s)$ can be analytically continued to an entire function. Observe that $E_{2n+1}^* = (-1)^n E_{2n}$, where n is a non-negative integer and E_{2j} denotes the $2j$th Euler number, which is defined by

$$\sec x = \sum_{j=0}^{\infty}\frac{(-1)^j E_{2j}x^{2j}}{(2j)!}, \quad |x| < \pi/2.$$

ENTRY 17

For each complex number r,

$$\varphi_r(-1/4)-\varphi_r(-3/4) = \frac{2\cos(\pi r/2)E_{r+1}^*}{4^{r+1}}.$$

Proof. Put $x = 1/4$ in Entry 15 to get, for $u > -1$,

$$\frac{\varphi_r(-3/4)-\varphi_r(-1/4)}{4\Gamma(r+1)} = -\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{(2\pi k)^{r+1}}$$

$$= -\frac{\cos(\pi r/2)}{(2\pi)^{r+1}}L(r+1)$$

$$= -\frac{\cos(\pi r/2)E_{r+1}^*}{4^{r+1}2\Gamma(r+1)},$$

by (55) and (54). The desired result now follows for $u > -1$. By analytic continuation, the proposed formula is valid for all r.

COROLLARY

For $u < 0$,

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^r = \frac{1}{2}\cos(\pi r/2) E_{r+1}^*.$$

Proof. By Entry 17 and (13), with $u < 0$,

$$\begin{aligned}\frac{2\cos(\pi r/2)E_{r+1}^*}{4^{r+1}} &= \varphi_r(-1/4) - \varphi_r(-3/4)\\ &= \sum_{k=1}^{\infty} \{(k-3/4)^r - (k-1/4)^r\}\\ &= 4^{-r} \sum_{k=0}^{\infty} (-1)^k (2k+1)^r,\end{aligned}$$

and the result follows.

ENTRY 18

For each complex number r,

$$\cos(\pi r/2) E_1^*{}_{-r} = 2L(r) = (\pi/2)^r E_r^*/\Gamma(r). \tag{56}$$

The equalities in (56) yield the functional equation of $L(r)$ [16, p. 69],

$$L(r) = \cos(\pi r/2)(\pi/2)^{r-1}\Gamma(1-r)L(1-r).$$

Proof. We present here Ramanujan's argument.
By (54), the "constant" for the series $L(r)$ is

$$L(r) = \frac{(\pi/2)^r}{2\Gamma(r)} E_r^*.$$

But by the last corollary, the "constant" for $L(r)$ is also equal to

$$\tfrac{1}{2}\cos(\pi r/2) E_1^*{}_{-r}.$$

Since these two constants must be equal, (56) follows at once.

Ramanujan's derivation of the next corollary was evidently very similar to his argument for Corollary 4 in section 4.

COROLLARY

We have

$$\pi\left(1/2 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^{1/2} + (2k+1)^{1/2}}\right) = L(3/2).$$

Proof. Since the proof is very similar to that of Corollary 4 in section 4, we shall present only a brief sketch. If we replace $f(x)$ by $f(x+\pi/2)$ in [47, pp. 153–154], we find that

$$\pi^{p+1}+\pi^{p+1}\sum_{k=1}^{\infty}(-1)^k\{(2k+1)^p-(2k-1)^p\}$$
$$=2^{p+2}\cos(\pi p/2)\Gamma(p+1)L(p+1),$$

where $0<p<1$, after a completely analogous argument. Putting $p=1/2$, we complete the proof of the Corollary.

ENTRY 19(i)

Assume the hypotheses of Corollary 16(i) with the additional assumption that q is odd. Then

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$
$$=\sin(\pi r/2)\sum_{j=1}^{(q-1)/2}\sin(2\pi jp/q)\left\{\varphi_{-r}\left(\frac{j}{q}-1\right)-\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. On the right side of (16.1) replace j by $q-j$ in that part of the sum with $(q+1)/2\leqslant j\leqslant q-1$.

ENTRY 19(ii)

Suppose that all hypotheses of Corollary 16(ii) hold. Assume also that q is odd. Then

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2\zeta(1-r)\right\}$$
$$=\cos(\pi r/2)\sum_{j=1}^{(q-1)/2}\cos(2\pi jp/q)\left\{\varphi_{-r}\left(\frac{j}{q}-1\right)+\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. Using (53), proceed in the same fashion as in the previous proof. In addition, the functional equation (18) must be employed.

Ramanujan's version of Entry 19(ii) is incorrect [44, vol. 2, pp. 86, 87].

COROLLARY 1

Let $u>0$ and suppose that $0\leqslant x<1$. Then

$$\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x)=\sum_{k=1}^{\infty}\frac{\sin(\pi kx)\cos(\pi kx+\pi r/2)}{k^{r+1}}.$$

Proof. By (14), (18) and (51), we find that for $u>0$ and $0\leqslant x<1$,

$$\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x)=\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\{\zeta(-r)-\zeta(-r,1-x)\}$$
$$=-\frac{1}{2}\sin(\pi r/2)\zeta(r+1)+\frac{1}{2}\sin(\pi r/2)$$
$$\times\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{r+1}}+\frac{1}{2}\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{r+1}}$$
$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{r+1}}\{-\sin(\pi r/2)+\sin(2\pi kx+\pi r/2)\}$$
$$=\sum_{k=1}^{\infty}\frac{\sin(\pi kx)\cos(\pi kx+\pi r/2)}{k^{r+1}},$$

upon using the identity $-\sin A + \sin(A+2B) = 2\sin B\cos(A+B)$. This completes the proof.

COROLLARY 2

If $0 < x < 1$, then

$$\sum_{k=0}^{\infty}\left(\frac{1}{(k+x)^{1/2}} - \frac{1}{(k+1-x)^{1/2}}\right) = 2\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{\sqrt{k}}.$$

Proof. Set $r = -1/2$ in Entry 15 and use (13).

ENTRY 20

If r is any complex number, then

$$\frac{(6\pi)^r}{2\sqrt{3}\Gamma(r)}\{\varphi_{r-1}(-1/3) - \varphi_{r-1}(-2/3)\}$$
$$= \sin(\pi r/2)\{\varphi_{-r}(-1/3) - \varphi_{-r}(-2/3)\}.$$

Proof. Put $p = 1$ and $q = 3$ in Entry 19(i), and the result follows.

Section 21 appears to have no relation to the other material in Chapter 7. In Entry 21, Ramanujan writes

$$\varphi_\infty\left(\frac{nx}{1+x}\right) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n}{k}\varphi(k)x^k, \tag{57}$$

where

$$\varphi_r(x) = \varphi_{r-1}(x) - 1 + \exp\left(\frac{nP_{r-1}\varphi_{r-1}^r(x)}{r!}\right),$$

with $\varphi_1(x) = \varphi(x)$, and where

$$P_r = \sum_{k=1}^{\infty}(-1)^{k+1}k^r x^k.$$

We have been unable to discern the meaning of this result, since the recursively defined functions $\varphi_r(x)$ have not been connected with (57) in any way. We shall regard (57) as the definition of φ_∞. Setting $u = nx/(1+x)$ and $p = u/n$, we find that (57) becomes

$$\varphi_\infty(u) = \sum_{k=0}^{\infty}\binom{n}{k}\varphi(k)p^k(1-p)^{n-k}. \tag{58}$$

In the following corollary, Ramanujan gives a formula for $\varphi_\infty(u)$ in terms of the derivatives $\varphi^{(j)}(u)$, $0 \leqslant j < \infty$. Note that $\varphi_\infty(u)$ is the expected value of $\varphi(u)$ if u denotes a random variable with binomial distribution $b(n, k; p)$. Ramanujan alludes to Entry 10 of Chapter 3 [4], where he gives a formula for the expected value of $\varphi(u)$ in terms of $\varphi^{(j)}(u)$, $0 \leqslant j < \infty$, where u denotes a Poisson random variable. However, the latter result appears to be considerably deeper than the present corollary.

COROLLARY

Let u and n be fixed where $0 \leqslant u \leqslant n$ and n is an integer. Let $\varphi(z)$ be analytic in a disc

centered at u and containing the segment $[0, n]$. If $\varphi_\infty(u)$ is defined by (58), then

$$\varphi_\infty(u) = \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} \sum_{k=0}^{\infty} \binom{n}{k} (k-u)^j p^k (1-p)^{n-k}. \tag{59}$$

Proof. Expanding $\varphi(z)$ in its Taylor series about u, we find that

$$\varphi_\infty(u) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} (k-u)^j p^k (1-p)^{n-k}.$$

The equality (59) now follows by inverting the order of summation.

Observe that the Corollary even holds when n is an arbitrary positive number, provided that $p < 1/2$ and φ is a polynomial. It would be interesting to find more general conditions under which (59) holds.

ENTRY 22

Let $A_1 = 0$. For each non-negative integer r, $r \neq 1$, set $A_r = \{1 + (-1)^r\}\zeta(r)$. If n is a natural number, then

$$\sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = \sum_{k=0}^{n} A_{n-k} \binom{-n}{k}.$$

Proofs of Entry 22 have been given by Glaisher [24], Kesava Menon [35], and Djoković [17]. Entry 22 is identical with Entry 35 in Chapter 9 [5]. The following example is the special case $n = 3$ of Entry 22.

EXAMPLE

$$\sum_{k=1}^{\infty} \frac{1}{k^3 (k+1)^3} = 10 - \pi^2.$$

Entry 23 offers the very well-known asymptotic expansion of $\log \Gamma(z+1)$ [50, p. 252].

ENTRY 23

Let $|\arg z| < \pi$. Then as $|z| \to \infty$.

$$\log \Gamma(z+1) \sim (z+1/2)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}}.$$

Ramanujan remarks that $\frac{1}{2}\log(2\pi) = 0{\cdot}918938533204673$, which is correct [1, p. 3].

We quote the following corollary exactly as it appears in the notebooks [44, vol. 2, p. 88]. This approximation for the gamma function is remindful of Corollaries 6–8 in section 4.

COROLLARY

When x is great $e^x \Gamma(x+1)/x^x = (2\pi x + \pi/3)^{1/2}$ nearly.

Proof. From the familiar asymptotic expansion for $\Gamma(x)$, as $x \to \infty$ [50, p. 253],

$$\frac{e^x \Gamma(x+1)}{x^x} \sim \sqrt{2\pi x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots\right). \tag{60}$$

But, on the other hand,

$$(2\pi x + \pi/3)^{1/2} = (2\pi x)^{1/2}\,(1 + 1/(6x))^{1/2}$$
$$= (2\pi x)^{1/2}\left(1 + \frac{1}{12x} - \frac{1}{288x^2} + \cdots\right). \tag{61}$$

Thus, Ramanujan's approximation is reasonable, but observe that the coefficients of x^{-2} in (60) and (61) are of opposite sign.

Entry 24 and its corollary are restatements of Corollaries 3 and 2, respectively, of section 4.

ENTRY 25

For every complex number z and positive integer n,

$$\prod_{k=1}^{n} \Gamma\left(\frac{z+k}{n}\right) = (2\pi)^{(n-1)/2}\, n^{-z-1/2}\, \Gamma(z+1).$$

Entry 25 is a version of Gauss's famous multiplication theorem for the gamma function [50, p. 240]. Corollary 1 is the special case $z = 0$ of Entry 25.

COROLLARY 2

$$\Gamma(2/3) = [\Gamma(5/6)]^{1/2}\, 2^{1/3}\, (\pi/3)^{1/4}.$$

Proof. Put $n = 2$ and $z = -1/3$ in Entry 25 to get $\Gamma(1/3)\Gamma(5/6) = \sqrt{\pi}\, 2^{1/3}\, \Gamma(2/3)$. By Corollary 3 of section 4, $\Gamma(2/3)\Gamma(1/3) = 2\pi/\sqrt{3}$. Combining these two equalities, we achieve the desired result.

COROLLARY 3

For every complex number z,

$$\Gamma(z+1) = \Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}+1\right) 2^z \pi^{-1/2}.$$

Corollary 3 is Legendre's duplication formula and is the special case $n = 2$ of Gauss's multiplication formula, Entry 25.

COROLLARY 4

Let $|\arg z| < \pi$. Then as $|z| \to \infty$,

$$\log \Gamma(z + 1/2) \sim z \log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}\,(2^{1-2k} - 1)}{2k(2k-1)z^{2k-1}}.$$

Proof. Replacing z by $2z$ in Corollary 3, we find that

$$\Gamma(z + 1/2) = \frac{\sqrt{\pi}\,\Gamma(2z+1)}{2^{2z}\Gamma(z+1)}.$$

Take logarithms on both sides and apply Entry 23.

Ramanujan inadvertently multiplied the infinite series above by -1.

The Maclaurin series in Entry 26 is well-known [1, p. 256].

ENTRY 26

For $|z| < 1$,

$$\log \Gamma(z+1) = -\gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)(-z)^k}{k},$$

where γ denotes Euler's constant.

COROLLARY

For $|z| < 1$,

$$\begin{aligned}\log\{\tfrac{1}{2}\Gamma(z+3)\} = {} & 0{\cdot}9227843351z + 0{\cdot}1974670334z^2 \\ & - 0{\cdot}0256856344z^3 + 0{\cdot}0049558084z^4 \\ & - 0{\cdot}0011355510z^5 + 0{\cdot}0002863437z^6 \\ & - 0{\cdot}0000766825z^7 + 0{\cdot}0000213883z^8 \\ & - 0{\cdot}0000061409z^9 + 0{\cdot}0000018013z^{10} + \ldots . \end{aligned} \tag{62}$$

Proof. Using Entry 26, we find that, for $|z| < 1$,

$$\begin{aligned}\log\{\tfrac{1}{2}\Gamma(z+3)\} &= \log(\tfrac{1}{2}z+1) + \log(z+1) + \log\Gamma(z+1) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{z}{2}\right)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}z^k}{k} \\ &\quad - \gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)(-z)^k}{k}.\end{aligned}$$

The given numerical values for the coefficients of z^k, $1 \leqslant k \leqslant 10$, now follow by direct calculation. Numerical values of $\zeta(k)$, $2 \leqslant k \leqslant 10$, may be found in [1, p. 811] or [18, p. 224].

In Ramanujan's formulation of (62), he replaces the tenth and all succeeding terms by the single expression $0{\cdot}0000054047\, z^{10}/(3+z)$.

Our calculations below were determined from the values of $\Gamma(2/3)$, $\Gamma(5/6)$, and $\Gamma(9/10)$ found in [21]. Ramanujan inexplicably gives the value 0·5341990853 for $\log \Gamma(2/3)$.

EXAMPLE 1

$$\log \Gamma(2/3) = 0{\cdot}3031502752.$$

EXAMPLE 2

$$\log \Gamma(5/6) = 0{\cdot}1211436313.$$

EXAMPLE 3

$$\log \Gamma(9/10) = 0{\cdot}0663762397.$$

ENTRY 27(i)

Suppose that n is a natural number and that $|z| > n$. Then

$$2\pi z \prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right) = 2\left(\frac{n!}{z^n}\right)^2 \sinh(\pi z)\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k\varphi_{2k}(n)}{kz^{2k}}\right). \tag{63}$$

Proof. Using Euler's definition of the gamma function [50, p. 237]

$$\Gamma(z) = \lim_{k\to\infty}\frac{(k-1)!\,k^z}{z(z+1)\dots(z+k-1)},$$

we find that

$$\begin{aligned}\frac{\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)} &= \lim_{k\to\infty}\frac{\{(n+1)^2+z^2\}\{(n+2)^2+z^2\}\dots\{(n+k)^2+z^2\}}{(n+1)(n+2)^2\dots(n+k)^2}\\ &= \prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right).\end{aligned} \tag{64}$$

On the other hand,

$$\begin{aligned}\frac{2\pi z\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)} &= \frac{2\pi z\Gamma^2(n+1)}{z^{2n}(1+1^2/z^2)(1+2^2/z^2)\dots(1+n^2/z^2)\Gamma(1+iz)\Gamma(1-iz)}.\end{aligned} \tag{65}$$

Using the Maclaurin series for $\log(1+y)$ with $y = k^2/z^2$, $1 \leqslant k \leqslant n$, we find that

$$\begin{aligned}\prod_{k=1}^{n}(1+k^2/z^2)^{-1} &= \exp\left(-\sum_{k=1}^{n}\log(1+k^2/z^2)\right)\\ &= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^j\varphi_{2j}(n)}{jz^{2j}}\right),\end{aligned} \tag{66}$$

provided that $|z| > n$. Also, by Corollary 3 of section 4,

$$\frac{2\pi z}{\Gamma(1+iz)\Gamma(1-iz)} = \frac{2\pi}{i\Gamma(iz)\Gamma(1-iz)} = 2\sinh(\pi z). \tag{67}$$

Now substitute (66) and (67) into (65). Comparing the resulting equality with (64), we readily deduce (63).

In Ramanujan's formulations of Entry 27 (i) and Entry 27 (ii) [44, vol. 2, pp. 89, 90], instead of $2\sinh(\pi x)$, there appears $e^{\pi x} - e^{-\pi x}\theta$, but $e^{-\pi x}\theta$ is struck out. In a footnote, which is also struck out, Ramanujan says that "$\theta = \cos 2\pi n$ exactly or very nearly according as $2n$ is an integer or not." A two line solution to Entry 27(i) is also crossed out.

ENTRY 27(ii)

Under the same hypotheses as Entry 27(i), we have

$$\begin{aligned}&2\pi(z^2+n^2)^{n+1/2}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right)\\ &= 2(n!)^2\sinh(\pi z)\exp\left(2n-2z\tan^{-1}(n/z)+\sum_{j=1}^{\infty}\frac{(-1)^jB_{2j}S_{2j}}{jz^{2j-1}}\right),\end{aligned} \tag{68}$$

where

$$S_{2j} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2j-1)!}{(2j-1)!(2k+1)!} \left(\frac{n}{z}\right)^{2k+1}, j \geqslant 1.$$

Proof. First,

$$(z^2+n^2)^{n+1/2}/z = z^{2n} \exp\left[(n+1/2)\log(1+n^2/z^2)\right]. \tag{69}$$

Multiplying both sides of (63) by $(z^2+n^2)^{n+1/2}/z$ and utilizing (69), we find that, for $|z| > n$,

$$2\pi(z^2+n^2)^{n+1/2} \prod_{k=1}^{\infty} \left(1+\frac{z^2}{(n+k)^2}\right)$$

$$= 2(n!)^2 \sinh(\pi z) \exp\left((n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}}\right). \tag{70}$$

Comparing (68) with (70), we see that it remains to show that

$$(n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}}$$

$$= 2n - 2z\tan^{-1}(n/z) + \sum_{j=1}^{\infty} \frac{(-1)^j B_{2j} S_{2j}}{jz^{2j-1}}. \tag{71}$$

By Entry 1 and the remarks prior to (8), since $\zeta(-2k) = 0$, $k \geqslant 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k+1}}{k(2k+1)z^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k}}{2kz^{2k}}$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{kz^{2k}} \sum_{j=1}^{k} \frac{B_{2j}(2k)! n^{2k-2j+1}}{(2j)!(2k-2j+1)!}. \tag{72}$$

Now a short calculation shows that

$$\sum_{k=1}^{\infty} \frac{(-1)^k n^{2k+1}}{k(2k+1)z^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k}}{2kz^{2k}} + (n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}}$$

$$= 2z \sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k+1}}{(2k+1)z^{2k+1}} = 2n - 2z\tan^{-1}(n/z).$$

Thus, we only need yet to examine the double sum in (72). Inverting the order of summation by absolute convergence, we find that this double series becomes

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \sum_{k=j}^{\infty} \frac{(-1)^k (2k)! n^{2k-2j+1}}{k(2k-2j+1)!z^{2k}}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j B_{2j}}{z^{2j-1}} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}(2\mu+2j)!}{(2j)!(\mu+j)(2\mu+1)!} \left(\frac{n}{z}\right)^{2\mu+1}$$

After a slight amount of simplification, the double series above is easily seen to be equal to the series on the right side of (71). This completes the proof of (71) and hence of (68).

ENTRY 27(iii).

Let n be a positive integer and suppose that $x > 0$. Write $r^2 = n^2 + x^2$ with $r > 0$ and

put $\beta = \tan^{-1}(x/n)$. Then as $x \to \infty$,

$$\log\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}$$
$$\sim 2\log\Gamma(n)+2n+2x\beta-\sum_{k=1}^{\infty}\frac{B_{2k}\cos\{(2k-1)\beta\}}{k(2k-1)r^{2k-1}}.$$

Proof. Using (64) and Entry 23, we find that, as $x \to \infty$,

$$\log\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}$$
$$=\log\left\{\frac{2\pi\{(n-1)!\}^2(x^2+n^2)^{n+1/2}}{\Gamma(n+1+ix)\Gamma(n+1-ix)}\right\}$$
$$=\log(2\pi)+2\log\Gamma(n)+(2n+1)\log r-\log\Gamma(n+1+ix)-\log\Gamma(n+1-ix)$$
$$\sim \log(2\pi)+2\log\Gamma(n)+(2n+1)\log r-(n+ix+1/2)\log(n+ix)$$
$$-(n-ix+1/2)\log(n-ix)-\log(2\pi)+2n$$
$$-\sum_{k=1}^{\infty}\frac{B_{2k}\{(n+ix)^{2k-1}+(n-ix)^{2k-1}\}}{2k(2k-1)r^{2(2k-1)}}.$$

The desired result now follows since $n+ix = re^{i\beta}$.

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Bounded starlike functions of complex order

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Abstract. Let $F(b, M)$ ($b \neq 0$ complex, $M > \frac{1}{2}$), denote the class of functions $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ analytic in $U = \{z: |z| < 1\}$ which satisfy for fixed $M, f(z)/z \neq 0$ in U and

$$\left| \frac{b - 1 + [zf'(z)/f(z)]}{b} - M \right| < M, \; z \in U.$$

In this note we obtain various representations for functions in $F(b, M)$. We maximize $|a_3 - \mu a_2^2|$ over the class $F(b, M)$. Also sharp coefficient bounds are established for functions in $F(b, M)$. We also obtain the sharp radius of starlikeness of the class $F(b, M)$.

Keywords. Analytic functions; starlike functions; bounded spiral-like functions; co-efficient estimate; radius of starlikeness.

1. Introduction

Let A denote the class of functions $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $\omega(z)$ in U satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \leqslant |z|$ for $z \in U$.

For $f \in A$, we say f belongs to the class $F(b, M)$ ($b \neq 0$, complex, $M > \frac{1}{2}$), of bounded starlike functions of complex order, if and only if $f(z)/z \neq 0$ in U and for fixed M,

$$\left| \frac{b - 1 + [zf'(z)/f(z)]}{b} - M \right| < M, \; z \in U. \tag{1}$$

One can easily show that $f \in F(b, M)$ if and only if there is a function $f_1 \in F(1, M)$ such that

$$f(z) = z[f_1(z)/z]^b. \tag{2}$$

It was shown in [4] that $f_1 \in F(1, M)$ if and only if for $z \in U$

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1 + \omega(z)}{1 - m\omega(z)}, \; \omega \in \Omega, \; m = 1 - 1/M. \tag{3}$$

Thus from (2) and (3) it follows that $f \in F(b, M)$ if and only if for $z \in U$

$$\frac{zf'(z)}{f(z)} = \frac{1 + \omega(z)[b(1 + m) - m]}{1 - m\omega(z)}, \; \omega \in \Omega, \; m = 1 - 1/M. \tag{4}$$

It is noticed that, by giving specific values to b and M we obtain the following

important subclasses studied by various authors in earlier works:

(i) $F(1, 1) \equiv$ class of functions $f(z)$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \ (z \in U)$$

studied by Singh [8].

(ii) $F(1, M) \equiv$ class of functions $f(z)$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)} - M\right| < M, \ M > \tfrac{1}{2}, \ (z \in U)$$

studied by Singh and Singh [9].

(iii) $F(\cos\lambda \exp(-i\lambda), M) \equiv F_\lambda, M, |\lambda| < \pi/2$, is the class of bounded spiral-like functions $f(z)$ satisfying the condition

$$\left|\frac{\exp(i\lambda) zf'(z)/f(z) - i\sin\lambda}{\cos\lambda} - M\right| M \quad (z \in U)$$

studied by Kulshrestha [4].

Obviously $F(\cos\lambda \exp(-i\lambda), \infty) = S^\lambda, |\lambda| < \pi/2$, is the class of λ-spirallike functions introduced by Špáček [10] and studied in [3] and [5]. Also $F(b, \infty) = S(b)$, is the class of starlike functions of complex order introduced by the authors [7]. Thus the results obtained here generalize the results due to [2], [4] to [9] and [11].

2. Coefficient estimates for the class $F(b, M)$

We need in our proof the following lemma.

LEMMA 1. (Keogh and Merkes [2]).

Let $\omega(z) = \Sigma_{n=1}^{\infty} c_n z^n \in \Omega$, if ν is any complex number, then

$$|c_2 - \nu c_1^2| \leqslant \max\{1, |\nu|\}. \tag{5}$$

Equality may be obtained with the functions $\omega(z) = z^2$ and $\omega(z) = z$.

THEOREM 1

If $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \in F(b, M)$, $M > 1$, and μ is any complex number, then

$$|a_3 - \mu a_2^2| \leqslant \frac{|b|(1+m)}{2m} \max\left\{1, \left|b\left(\frac{1+m}{m}\right)(2\mu - 1) - 1\right|\right\}. \tag{6}$$

This inequality is sharp for each μ.

Proof. Since $f \in F(b, M)$ we get from (4) after expanding and equating the coefficients

$$a_2 = b(1+m)\, c_1, \tag{7}$$

$$a_3 = \frac{b(1+m)}{2}\{[b(1+m)+m]c_1^2 + c_2\}. \tag{8}$$

Using (5), (7) and (8) we get the result. Since (5) is sharp, (6) is also sharp.

COROLLARY 1

If $f \in F(b, M)$, $M > 1$, then

$$|a_2| \leqslant |b|(1+m), \tag{9}$$

$$|a_3| \leqslant \frac{|b|(1+m)}{2m} \max\left\{1, \left|b\left(\frac{1+m}{m}\right)+1\right|\right\}. \tag{10}$$

Proof. The inequalities (9) and (10) follow directly from (7) and (6), respectively.

THEOREM 2

If $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \in F(b, M)$, $|b|^2(1+m)+2m(k-1)\ \mathrm{Re}\{b\} + (m-1)(k-1)^2 > 0$. Then for $n = 2, 3, \ldots$.

$$|a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=0}^{n-2} |b(1+m)+mk|, \quad m = 1 - 1/M. \tag{11}$$

The bounds in (11) are sharp for the function

$$f_0(z) = \begin{cases} z/(1-mz)^{b((1+m)/m)} & \text{for } m \neq 0 \\ z \exp\{bz\} & \text{for } m = 0 \end{cases} \tag{12}$$

Proof. For $f \in F(b, M)$, we have from (4) that

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)[b(1+m)-m]}{1-m\omega(z)}, \quad \omega \in \Omega. \tag{13}$$

Following Clunie [1], we set

$$\omega(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{14}$$

Therefore from (13) and (14), we have

$$\left[b(1+m)z + \sum_{k=2}^{n-1} (b(1+m)+m(k-1))\, a_k z^k\right] \omega(z)$$

$$= \sum_{k=2}^{n} (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k,$$

where d_ks are some complex numbers. Then since $|\omega(z)| < 1$, we have

$$\left|b(1+m)z + \sum_{k=2}^{n-1} (b(1+m)+m(k-1)) a_k z^k\right|$$

$$\geqslant \left|\sum_{k=2}^{n} (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k\right|. \tag{15}$$

Squaring both sides of (15) and integrating round $|z| = r < 1$, we get, after taking the limit when $r \to 1$,

$$(n-1)^2 |a_n|^2 \leqslant |b(1+m)|^2 + \sum_{k=2}^{n-1} [|b(1+m)$$

$$+ m(k-1)|^2 - (k-1)^2] |a_k|^2 \tag{16}$$

and since $|b|^2(1+m)+2m(k-1)\,\mathrm{Re}\{b\}+(m-1)(k-1)^2>0$, (16) gives successively for $n=2, 3, \ldots$

$$|a_2|\leqslant|b(1+m)|,\quad |a_3|\leqslant\frac{|b(1+m)||b(1+m)+m|}{2!},$$

$$|a_4|\leqslant\frac{|b(1+m)||b(1+m)+m||b(1+m)+2m|}{3!},\ldots.$$

and hence by induction

$$|a_n|\leqslant\frac{1}{(n-1)!}\prod_{k=0}^{n-2}|b(1+m)+mk)|,\quad n=2, 3, \ldots, m=1-1/M.$$

This completes the proof of the theorem.

COROLLARY 2

If $f(z)=z+\Sigma_{n=2}^{\infty}a_n z^n\in F(b, M)$,

$$|b|^2(1+m)+2m(k-1)\ \mathrm{Re}\ \{b\}+(m-1)(k-1)^2\leqslant 0.$$

Then for $n\geqslant 2$

$$|a_n|\leqslant\frac{|b(1+m)|}{(n-1)}.$$

This result is sharp.

Remark: (i) For $b=\cos\lambda\exp(-i\lambda)$, $|\lambda|<\pi/2$, and $m=1$, the theorems 1 and 2 are proved respectively in [2], [5] and [11].

(ii) For $b=1$ and $m=0$, theorem 2 is proved by Singh [8].

(iii) For $b=1$, theorem 2 is proved in [9].

(iv) For $b=\cos\lambda\exp(-i\lambda)$, $|\lambda|<\pi/2$, theorem 2 is proved in [4].

(v) For $m=1$, the theorems 1 and 2 are proved in [7].

3. Univalence

LEMMA 2

If $f\in F(b, M)$, $a\in U$, then

$$F(z)=\frac{maz\cdot f(z+ma/1+m\bar{a}z)}{f(ma)(z+ma)(1+m\bar{a}z)^{((1+m)/m)-1}},\ z\in U,\tag{17}$$

is likewise in $F(b, M)$, $M>1$.

Proof. Since $f\in F(b, M)$, we have

$$f(z)=z[f_1(z)/z]^b, f_1\in F(1, M).\tag{18}$$

It is well-known [4], that if $f_1(z)\in F(1, M)$, then $F_1(z)$ defined by $F_1(0)=0$ and

$$F_1(z)=\frac{maz\cdot f_1(z+ma/1+m\bar{a}z)}{f_1(ma)(z+ma)(1+m\bar{a}z)^{1/m}},\ z\in U,\tag{19}$$

belongs to $F(1, M)$, $M > 1$.

Therefore it follows from (18) and (19) that the function $F(z)$ defined by $F(0) = 0$ and

$$F(z) = z\left[\frac{F_1(z)}{z}\right]^b = \frac{maz \cdot f(z+ma/1+m\bar{a}z)}{f(ma)(z+ma)(1+m\bar{a}z)^{b((1+m)/m)-1}}, \quad z \in U,$$

also belongs to $F(b, M)$, $M > 1$, which proves the lemma.

LEMMA 3

For $|z| \leqslant r$ and f ranging over $F(b, M)$, $M > 1$, then the domain of values of $zf'(z)/f(z)$ is the disc with center

$$\left(\frac{1+[\mathrm{Re}\{b\}(1+m/m)-1]r^2}{1-r^2}, \quad \frac{\mathrm{Im}\{b\}(1+m/m)r^2}{1-r^2}\right)$$

and radius $\dfrac{|b|(1+m)(r)}{1-r^2}$.

Proof. Suppose $f(z) \in F(b, M)$. Then by Lemma 2, $F(z)$, defined by (17) is also in $F(b, M)$, $M > 1$. Hence

$$\left|\frac{F''(0)}{2}\right| = \left|\frac{f'(ma)}{f(ma)}(1-m^2|a|^2) - \frac{1+[b(1+m/m)-1]m^2|a|^2}{ma}\right|$$

$$\leqslant |b|(1+m), \text{ from (9), for } |a| < 1.$$

Hence

$$\left|\frac{f'(ma)}{f(ma)} - \frac{1+[b(1+m/\mathrm{m})-1]m^2|a|^2}{ma(1-m^2|a|^2)}\right| \leqslant \frac{|b|(1+m)}{(1-m^2|a|^2)}.$$

Replacing ma by z in the above inequality, we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+[b(1+m/m)-1]r^2}{1-r^2}\right| \leqslant \frac{|b|(1+m)r}{1-r^2}, \tag{20}$$

and the proof is completed.

THEOREM 3

The sharp radius of starlikeness of the class $F(b, M)$, $M > 1$, is

$$r_s = 2\{|b|(1+m)+[|b|^2(1+m)^2-4(\mathrm{Re}\{b\}(1+m/m)-1)]^{1/2}\}^{-1}. \tag{21}$$

Proof. From Lemma 3, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1-|b|(1+m)r+[\mathrm{Re}\{b\}(1+m/m)-1]r^2}{1-r^2}.$$

Thus $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < r_s$ where r_s is given by (21).

To show that (21) is sharp, we let

$$f_0(z) = z(1-mz)^{-b(1+m/m)} \text{ and } t = \frac{r[r-m(\bar{b}/b)^{1/2}}{m[1-mr(\bar{b}/b)^{1/2}]}$$

and obtain

$$\frac{tf_0'(t)}{f_0(t)} = \frac{1-|b|(1+m)r+[b(1+m/m)-1]r^2}{1-r^2},$$

which has the real part of value zero at r given by (21). This completes the proof of the theorem.

Remark: (i) For $b = \cos\lambda \exp(-i\lambda)$, $|\lambda| < \pi/2$, and $m = 1$, the lemma 2 and 3 and theorem 3 are proved in [6].

(ii) For $b = \cos\lambda \exp(-i\lambda)$, $|\lambda| < \pi/2$, the lemmas 2 and 3 and theorem 3 are proved in [4].

(iii) For $m = 1$, the lemmas 2 and 3 and theorem 3 are proved in [7].

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Monotonicity of polymatroids

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Abstract. Unlike matroids, it is observed that there is an upper bound for the number k such that the polymatroid is k-monotone. We define this upper bound as the monotonicity of polymatroid. We aim to calculate the value of this quantity in terms of the ground rank function of the polymatroid.

Keywords. Polymatroids; Knapsack problems; monotonicity.

1. Introduction

Monotonic properties of a matroid and of an independent system have been studied recently by several authors ([1], [2]). It has been established that if a matroid is 3-monotone then it is also k-monotone for any other integer k. The aim of the present paper is to explore the monotonic property of a polymatroid. It is observed that in polymatroids there is no number t such that t-monotonicity would imply k-monotonicity for any $k > t$. We therefore define monotonicity of a polymatroid as a number μ such that the polymatroid is k-monotone if and only if $k \leqslant \mu$. In this paper we determine the value of μ in terms of the ground rank function.

2. Definition and preliminaries

Let E be a finite set and ρ be a function from the set 2^E of all subsets of E into the set R_+ of all nonnegative real numbers satisfying

$$\text{(i)} \quad \rho(\phi) = 0, \tag{1}$$

$$\text{(ii)} \quad \rho(A) \leqslant \rho(B) \text{ for all } A \subseteq B \subseteq E, \tag{2}$$

$$\text{(iii)} \quad \rho(A) + \rho(B) \geqslant \rho(A \cup B) + \rho(A \cap B), \tag{3}$$

$$\text{for all } A, B \subseteq E.$$

The pair (E, ρ) is called a polymatroid. We denote R_+^E the set of all vectors $x = (x(e))_{e \in E}$ with nonnegative components. For any $x \in R_+^E$ we let

$$x(A) = \sum_{e \in A} x(e) \qquad \text{for all } A \subseteq E,$$

and $$|x| = x(E).$$

A vector $x \in R_+^E$ is called an independent vector of (E, ρ) if $x(A) \leqslant \rho(A)$ for all $A \subseteq E$.

P denotes the set of all independent vectors of (E, ρ). The union of x and y, denoted by $x \cup y$, has coordinates max $(x(e), y(e))$ for all $e \in E$. We define an order $\leqslant$ in R_+^E as $x \leqslant y \Leftrightarrow x(e) \leqslant y(e)$ for all $e \in E$.

Another order relation $\geqslant$ is defined in R_+^E as follows. If $x \cup z \in P \Rightarrow y \cup z \in P$ for all $z \in P$ then we say $x \geqslant y$.

A polymatroid is called k-monotone if for each pair x, y such that $|x \cup y| \leqslant k$ either $x \geqslant y$ or $y \geqslant x$. It is clear that if (E, ρ) is k-monotone then it is also k'-monotone for $k' \leqslant k$. Monotonicity of (E, ρ) is the number μ such that (E, ρ) is k-monotone if and only if $k \leqslant \mu$. Here we concentrate on finding the value of μ in terms of ρ.

Let us assume that the elements of E are arranged so that $\theta(e_1) \leqslant \theta(e_2) \leqslant \ldots \leqslant \theta(e_n)$ where $\theta(e) = \rho(E) - \rho(E - (e))$ for all $e \in E$. We shall prove that $\mu \geqslant \theta(e_1) + \theta(e_2)$.

Let $x^* \in R_+^E$ be defined as $x^*(e) = \theta(e)$ for all $e \in E$.

3. Monotonicity

Let us first prove a result characterizing the element whose union with a given element x is independent.

THEOREM 1.

Given $x \in P$, $x \cup z \in P$ for $z \in R_+^E$ if and only if

$$z(B) \leqslant \min_{A \supseteq B} \{\rho(A) - x(A - B)\} \quad \text{for all} \quad B \subseteq E \tag{4}$$

Proof: Let $y = x \cup z$. So $x \cup z \in P$ if and only if

$$y(A) \leqslant \rho(A) \quad \text{for all} \quad A \subseteq E. \tag{5}$$

Clearly, from the definition of $x \cup z$, we have

$$z(B) + x(A - B) \leqslant y(A) \quad \text{for all} \quad B \subseteq A$$

Now if $x \cup z \in P$ then

$$z(B) + x(A - B) \leqslant \rho(A) \text{ for all } B \subseteq A \tag{6}$$

$$z(B) \leqslant \rho(A) - x(A - B), \text{ for all } B \subseteq A \tag{7}$$

Since (7) is true for all supersets A for a given B, we have

$$z(B) \leqslant \min_{A \supseteq B} \{\rho(A) - x(A - B)\}.$$

Conversely, if (4) holds then (7) holds and so does (6). Hence

$$\rho(A) \geqslant \max_{B \subseteq A} \{z(B) + x(A - B)\} \text{ for all } A \subseteq B = y(A), \text{ for all } A \subseteq E.$$

COROLLARY 1.

For any $x \in P$ such that $x(e) \leqslant \theta(e)$ for all $e \in E$,

$$x \cup z \in P \text{ if and only if } z \in P$$

Proof:

$$\rho(A) - \theta(A-B) = \rho(A) - \sum_{e \in A-B} [\rho(E) - \rho(E-e)]$$

$$= \rho(A) - \gamma\rho(E) + \sum_{e \in A-B} \rho(E-e), \gamma = |A-B|$$

$$= \rho(A) - \rho(E) + \rho(E-f) - (\gamma-1)\rho(E) + \sum_{\substack{e \in A-B \\ e \neq f}} \rho(E-e),$$

for any $f \in A-B$

$$\geqslant \rho(A-f) - (\gamma-1)\rho(E) + \sum_{\substack{e \in A-B \\ e \neq f}} \rho(E-e) \quad \text{by} \quad (3)$$

$$= \rho(A-f) - \sum_{\substack{e \in A-B \\ e \neq f}} [\rho(E) - \rho(E-e)]$$

$$= \rho(A-f) - \theta(A-B-f).$$

Applying this repeatedly for f in $A-B$, we get $\rho(A) - \theta(A-B) \geqslant \rho(B)$. Hence by hypothesis $\rho(A) - x(A-B) \geqslant \rho(B)$. But for $A = B$, $x(A-B) = 0$ and $\rho(A) = \rho(B)$. Thus $\min_{A \supseteq B} \{\rho(A) - x(A-B)\} = \rho(B)$ for all B.

From (4) $z(B) \leqslant \rho(B)$ becomes the required condition for $x \cup z \in P$ and this condition holds if $z \in P$. Hence, the theorem is proved.

The above two results identify the values of z so that $x \cup z$ is independent. Making use of these results we shall identify the relation between any two elements z and y with respect to the ordering $\geqslant$.

COROLLARY 2.

For any x, y such that $x \leqslant x^*$ and $y \leqslant x^*$ the following are true simultaneously

(i) $x \geqslant y$ (ii) $y \geqslant x$.

Proof: It is clear from corollary 1 that if $x \leqslant x^*$ then $x \cup z \in P$ for $z \in P$. Similarly $y \cup z \in P$ for $z \in P$.

Now we shall prove the first part of the main result of the paper.

THEOREM 2.

For any x, $y \in P$ such that $|x \cup y| \leqslant \theta(e_1) + \theta(e_2)$ then either $x \geqslant y$ or, $y \geqslant x$.

Proof: We shall separately analyse three different cases.

Case (*i*) Both $x \leqslant x^*$ and $y \leqslant x^*$,

Theorem follows from corollary 2.

Case (*ii*) $x < x^*$ or $y < x^*$

Let us assume that $x \leqslant x^*$ and $y \nleqslant x^*$.

If $y \cup z \in P$ then $z \in P$ and so $x \cup z \in P$. Thus $y \geqslant x$. Similar argument follows that $x \geqslant y$ if $y \leqslant x^*$ but $x \nleqslant x^*$.

Case (iii) $x \nleqslant x^*$ and $y \nleqslant x^*$,

Let us define

$$A_x = \{e \in E : x(e) > x^*(e)\} \text{ and } A_y = \{e \in E : y(e) > x^*(e)\}.$$

If $|x \cup y| \leqslant \theta(e_1) + \theta(e_2)$ then $|x| \leqslant \theta(e_1) + \theta(e_2)$, $|y| \leqslant \theta(e_1) + \theta(e_2)$

Hence A_x cannot have more than one element and so does A_y. We assume that $A_x \neq A_y$.

So if $i \in A_x$ and $j \in A_y$ then $\max\{x(i), y(i)\} = x(i)$

$$\max\{x(j), y(j)\} = y(j) \text{ and } x(i) + y(j) > \theta(e_1) + \theta(e_2).$$

Hence $|x \cup y| > \theta(e_1) + \theta(e_2)$, which is a contradiction. So we have $A_x = A_y$. Without loss of generalities we assume that $y(i) \geqslant x(i)$.

We shall show that $y \geqslant x$. We know that for any z with $y \cup z \in P$ we have $z(B) \leqslant \min_{A \supseteq B} \{\rho(A) - y(A - B)\}$ for all $B \subseteq E$. Whenever $i \in B$, $i \notin A - B$ and so

$$y(e) \leqslant x^*(e) = \theta(e) \text{ for } e \in A - B.$$

So $\rho(A) - y(A - B) \leqslant \rho(B)$ for all B containing i.

In other words, for each B

$$\min_{A \supseteq B} \{\rho(A) - y(A - B)\} = \rho(B).$$

If $i \notin B$, then for those A such that $i \notin A \supseteq B$,

$$\rho(A) - y(A - B) \leqslant \rho(B)$$

and for those A such that $i \in A \supseteq B$,

$$\rho(A) - y(A - B) \leqslant \rho(B \cup \{i\}) - y(i).$$

Hence, $z \cup y \in P$

$$\Leftrightarrow z(B) \leqslant \min\{\rho(B), \rho(B \cup \{i\}) - y(i)\}.$$

But we know that $x(i) \leqslant y(i)$.

Hence $z \cup y \in P$

$$\Rightarrow z(B) \leqslant \min\{\rho(B), \rho(B \cup \{i\}) - x(i)\}$$

$$\Leftrightarrow z \cup x \in P \text{ (retracing the same argument)}$$

So $y \geqslant x$ and this completes the proof.

Thus we have shown that

$$\mu \geqslant \theta(e_1) + \theta(e_2).$$

Let us now consider the following example.

Let $E = \{e_1, e_2, e_3\}$. The ground rank function ρ is defined as,

$$\rho(e_1) = 1, \rho(e_2) = 2, \rho(e_3) = 3, \rho(e_1, e_2) = 2,$$

$$\rho(e_1, e_3) = 3{\cdot}5, \rho(e_2, e_3) = 4{\cdot}5, \rho(e_1, e_2, e_3) = 5.$$

Let us take two vectors x and y as

$$x = (0{\cdot}5 + \varepsilon, 0{\cdot}4, 0{\cdot}0) \qquad \text{and } y = (0{\cdot}4, 1{\cdot}5 + \varepsilon, 0{\cdot}0)$$

for any $\varepsilon > 0$.

The values of θ can be computed from the definition.

$$\theta(e_1) = 0{\cdot}5, \theta(e_2) = 1{\cdot}5, \theta(e_3) = 2{\cdot}5$$

$$u = \theta(e_1) + \theta(e_2) = 2.$$

$$|x \cup y| = 2 + 2\varepsilon > \mu.$$

We know that any element z such that $z \cup x \in P$ must satisfy

$$z(e_1) \leqslant 1, z(e_2) \leqslant 2 - \varepsilon, z(e_3) \leqslant 3 - \varepsilon,$$

$$z(e_1 + e_2) \leqslant 2{\cdot}5, z(e_1, e_3) \leqslant 3{\cdot}5, z(e_2, e_3) \leqslant 4{\cdot}5 - E$$

$$z(e_1, e_2, e_3) \leqslant 5.$$

Any element v with $v \cup y \in P$ must satisfy

$$v(e_1) \leqslant 1 - e \quad v(e_2) \leqslant 2 \quad v(e_3) \leqslant 3 - \varepsilon$$

$$v(e_1, e_2) \leqslant 2{\cdot}5 \quad v(e_1, e_3) \leqslant 3{\cdot}5 - \varepsilon$$

$$v(e_2, e_3) \leqslant 4{\cdot}5 \quad v(e_1, e_2, e_3) \leqslant 5.$$

Thus if $z = (1 - \varepsilon/2, 1{\cdot}3, 2{\cdot}45)$

$$\text{v} = (0{\cdot}3, 2 - \varepsilon/2, 1) \text{ with } 0 < \varepsilon < 0{\cdot}1$$

then $\quad x \cup z \in P$ but $y \cup z \notin P \Rightarrow x \not\geqslant y$

also $\quad v \cup y \in P$ but $v \cup x \notin P \Rightarrow y \not\geqslant x.$

Hence for any value of $0 < \varepsilon < 0{\cdot}1$, x and y are not comparable with respect to the ordering $\geqslant$. Hence we conclude that if $k > \mu$ then P is not k-monotone.

Thus we prove the inequality $\mu \geqslant \theta(e_1) + \theta(e_2)$ for all polymatroids and that there exists a polymatroid where it is met with equality.

4. Conclusion

Monotonicity for matroids has been studied in the context of threshold properties of the independent sets. The latter, in its turn, is useful in defining valid inequalities of Knapsack problems. Although there is no such motivation to study monotonicity for polymatroids it is observed that such a study would be very useful in solving problems such as multiterminal flow and sharing problems of network.

It is also observed that the order relation $\geqslant$ is an equivalence relation in P. In case of matroids such equivalence relation has been studied in the context of series reduction. This motivates one to define such concepts in a polymatroid. We propose to pursue this line in our next research.

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Surface waves in micropolar thermoelasticity

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Abstract. Free surface waves of arbitrary form in a homogeneous and isotropic linear micropolar thermoelastic half-space with stress-free plane boundary are investigated. It is found that all physical quantities associated with the waves are derivable from two scalar functions and that there exist two families of waves in general. One of these is the classical thermoelastic wave modified under the influence of the microelastic field and the other is a new surface wave not encountered in classical elasticity. The waves are not necessarily plane waves and even when these are assumed to propagate in a fixed direction parallel to the boundary, unlike in classical elasticity, the problem is not one of plane strain. Explicit expressions for the displacement vector, microrotation vector and the temperature are obtained and the nature of deformation has been analysed. Several earlier results are deduced as particular cases of the more general results obtained here.

Keywords. Micropolar thermoelasticity; surface waves of arbitrary form; couple stress; microrotation; generalized thermoelasticity.

1. Introduction

The theory of micropolar elasticity introduced and developed by Eringen [9–11] has aroused much interest in recent years because of its possible utility in investigating deformation properties of solids for which the classical (Hookean) theory is inadequate. This theory is believed to be particularly useful in investigating materials consisting of bar-like molecules which exhibit microrotational effects and which can support body and surface couples. Further, the micropolar elastic model is considered to be more realistic than the classical elastic model in studying geophysical problems. Several investigations revealing interesting phenomena which characterize the micropolar theory and its generalizations are contained in references [3, 8–13, 15, 19, 20].

In this paper we study free surface waves in a homogeneous and isotropic linear micropolar thermoelastic half-space with stress-free plane boundary. This problem is of geophysical interest, particularly in investigations concerned with earthquakes, and we study the problem by employing the basic equations obtained in [20]. We analyse the surface waves without making any assumption regarding the form of the wavefront and find that all field variables associated with the waves are derivable from two scalar functions. There exist, in general, two families of waves; one of these families is the classical thermoelastic surface wave influenced by the micropolar field and the other family is a new surface wave not encountered in classical elasticity. These waves, unlike the Rayleigh-type waves, are not necessarily plane waves and the secular equation holds for all types of surface waves, plane or otherwise. In the case of weak coupling between the elastic and micropolar fields, the wave propagation takes place in four families and in the limiting case of vanishingly small coupling, two of these four families disappear

and there remain just two uncoupled waves: the classical thermoelastic surface wave and the pure micropolar surface wave. We obtain explicit expressions for the displacement vector, microrotation vector and the temperature, and find that the displacement vector and the microrotation vector at a point of the body lie in two different planes and the orientation of these planes relative to the boundary as well as their relative inclination vary from point to point in the body. This result emphasises the necessity of treating the problem of surface waves as a three-dimensional one, rather than a two-dimensional one as is done in classical works concerned with Rayleigh-type waves.

The study of surface waves of arbitrary form was initiated by Knowles [16], who considered a classical (non-polar) elastic half-space for discussion. The present author extended this work to an elastic half-space with material boundary [6]. He also studied interfacial waves of arbitrary form at the plane interface between an elastic half-space and a non-viscous, compressible liquid [7]. Carroll [2] and Agarwal [1] extended the work of Knowles to coupled and generalized theories of thermoelasticity. In all these studies, it has been found that a *single* scalar function determines all physical quantities associated with the waves and that if the waves are assumed to propagate in a fixed direction parallel to the boundary (or an interface, as the case may be), the problem reduces to one of plane strain. This result is no longer valid in the present work. Indeed, we find that the physical quantities associated with the waves considered here depend on *two* scalar functions and that even when the waves are assumed to propagate in a fixed direction parallel to the boundary, the problem is *not* one of plane strain, in general. This deviation in the results between the studies referred to above and the present work is due to the presence of microrotational effects in the present work. However, if the microrotational effects are vanishingly small, our problem reduces to that considered in [2] and we recover the results obtained there. Further, if the effects of the thermal field are also vanishingly small, our problem and the corresponding results reduce to those of [16]. At the end of the paper, we show how the work of Agarwal [1] also follows as a particular case of our analysis.

In [19], Suhubi and Eringen studied *plane* surface waves in a non-thermal micropolar elastic half-space. Their analysis was based on the hypothesis that the displacement vector lies completely in the plane determined by the direction of wave propagation and the normal to the boundary and that the microrotation vector lies perpendicular to this plane. Our analysis here shows that this hypothesis is not generally valid and that the analysis presented in [19] is of very restricted nature. We exhibit the occurrence of a family of waves not encountered in [19] and show how the analysis presented there follows as a particular case of our analysis here.

2. Basic equations and boundary conditions

The equations governing the displacement vector $\mathbf{u}$, the microrotation vector $\boldsymbol{\phi}$ and the temperature change T in a homogeneous and isotropic linear micropolar thermoelastic body, in the absence of body forces, body couples and internal heat sources, are [20]

$$(\mu+\kappa)\nabla^2\mathbf{u}+(\lambda+\mu)\,\text{grad div}\,\mathbf{u}+\kappa\,\text{curl}\,\boldsymbol{\phi}$$

$$-(3\lambda+2\mu+\kappa)\,a\,\text{grad}\,T=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

$$\gamma\nabla^2\boldsymbol{\phi} + (\alpha+\beta)\,\text{grad div}\,\boldsymbol{\phi} + \kappa\,\text{curl}\,\mathbf{u} - 2\kappa\boldsymbol{\phi} = I\frac{\partial^2\boldsymbol{\phi}}{\partial t^2}, \tag{2}$$

$$k_1\nabla^2 T = \rho c_E\frac{\partial T}{\partial t} + (3\lambda+2\mu+\kappa)aT_0\frac{\partial}{\partial t}(\text{div}\,\mathbf{u}). \tag{3}$$

In these equations, λ and μ are the isothermal Lamé constants, α, β, γ and κ are the isothermal micropolar elastic constants, ρ and I are respectively the mass and the micro-inertia per unit volume, a and c_E are respectively the volume coefficients of thermal expansion and specific heat at constant strain and k_1 and T_0 are respectively the thermal conductivity and the temperature of the body in its initial (unstrained) state.

If we set

$$\left.\begin{aligned}\mathbf{u} &= \text{grad}\,p + \text{curl}\,\mathbf{q}\\ \text{and}\qquad \boldsymbol{\phi} &= \text{grad}\,\xi + \text{curl}\,\boldsymbol{\psi},\end{aligned}\right\} \tag{4}$$

equations (1) and (2) are satisfied if p, $\mathbf{q}$, ξ and $\boldsymbol{\psi}$ are governed by the following equations.

$$\left(\nabla^2 - \frac{1}{\alpha_1^2}\frac{\partial^2}{\partial t^2}\right)p = \frac{(3\lambda+2\mu+\kappa)aT}{(\lambda+2\mu+\kappa)}, \tag{5}$$

$$\left(\nabla^2 - \frac{1}{\alpha_2^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{q} = -\zeta\,\text{curl}\,\boldsymbol{\psi}, \tag{6}$$

$$\left(\nabla^2 - \frac{1}{\alpha_3^2}\frac{\partial^2}{\partial t^2} - 2\nu\right)\xi = 0, \tag{7}$$

$$\left(\nabla^2 - \frac{1}{\alpha_4^2}\frac{\partial^2}{\partial t^2} - 2\eta\right)\boldsymbol{\psi} = -\eta\,\text{curl}\,\mathbf{q}. \tag{8}$$

In these equations we have put

$$\left.\begin{aligned}&\alpha_1^2 = (\lambda+2\mu+\kappa)/\rho, \quad \alpha_2^2 = (\mu+\kappa)/\rho, \quad \alpha_3^2 = (\alpha+\beta+\gamma)/I, \quad \alpha_4^2 = \gamma/I,\\ &\zeta = \kappa/(\mu+\kappa), \quad \nu = \kappa/(\alpha+\beta+\gamma), \quad \eta = \kappa/\gamma\end{aligned}\right\} \tag{9}$$

Note that p and T are coupled together, $\mathbf{q}$ and $\boldsymbol{\psi}$ are coupled together and ξ is uncoupled with other potentials. If $\kappa = 0$, $\mathbf{q}$ and $\boldsymbol{\psi}$ become uncoupled. Then (3), (5) and (6) govern the classical coupled thermoelastic field and (7) and (8) govern the pure micropolar field. In this case ξ and $\boldsymbol{\psi}$ are solutions of wave equations associated with phase velocities α_3 and α_4 respectively. In what follows we assume that $\kappa \neq 0$, unless otherwise stated.

Eliminating T from (3) and (5) and using (4), we obtain

$$\left(\nabla^2 - \frac{1}{k_2}\frac{\partial}{\partial t}\right)\left(\nabla^2 - \frac{1}{\alpha_1^2}\frac{\partial^2}{\partial t^2}\right)p = \frac{\varepsilon_T\varepsilon_M}{k_2}\frac{\partial}{\partial t}(\nabla^2 p), \tag{10}$$

where we have put

$$\begin{aligned}&k_2 = k_1/\rho c_E, \quad \varepsilon_T = (3\lambda+2\mu)^2 a^2 T_0/(\lambda+2\mu)\rho c_E,\\ &\varepsilon_M = \left(1+\frac{\kappa}{3\lambda+2\mu}\right)^2\Big/\left(1+\frac{\kappa}{\lambda+2\mu}\right).\end{aligned} \tag{11}$$

Note that ε_T is the thermoelastic coupling constant of the coupled thermoelasticity ([18], p. 41).

From (5) to (8) and (10) it is clear that of the five unknowns $p, \xi, T, \mathbf{q}$ and ψ, only three, *viz.*, p, ξ and ψ (or $\mathbf{q}$) are independent. If in a given problem p, ξ and ψ are determined, T follows from (5) and $\mathbf{q}$ from (6) or (8).

Suppose the body occupies the domain $x_3 > 0$ referred to the Cartesean coordinate system x_i. If the boundary $x_3 = 0$ is traction-free and is subjected to linearized heat transfer condition, we have on $x_3 = 0$,

$$\tau_{3j} = m_{3j} = 0, \quad j = 1, 2, 3 \tag{12}$$

and

$$\frac{\partial T}{\partial x_3} + kT = 0, \tag{13}$$

where τ_{ij} and m_{ij} are respectively the components of the (force) stress and couple stress tensors and k is a non-negative constant. If $k = 0$, the boundary $x_3 = 0$ is thermally insulated and if $k \to \infty$, it is an isothermal boundary.

In the notation of Cartesean tensors, τ_{ij} and m_{ij} are given by [20]

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + \kappa(u_{j,i} - \varepsilon_{ijk}\phi_k) - (3\lambda + 2\mu + \kappa)aT\delta_{ij} \tag{14}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i}. \tag{15}$$

The boundary conditions (12) may be rewritten and expressed in terms of u_i and ϕ_i in the following way, on using (14) and (15).

$$\begin{aligned}
&\tau_{33} \equiv \lambda u_{k,k} + (2\mu + \kappa)u_{3,3} - (3\lambda + 2\mu + \kappa)aT = 0,\\
&\tau_{3\alpha,\alpha} \equiv \mu u_{3,\alpha\alpha} + (\mu + \kappa)u_{\alpha,\alpha 3} + \kappa\varepsilon_{\alpha\beta}\phi_{\alpha,\beta} = 0,\\
&\varepsilon_{\alpha\beta}\tau_{3\alpha,\beta} \equiv (\mu + \kappa)\varepsilon_{\alpha\beta}u_{\alpha,\beta 3} - \kappa\phi_{\alpha,\alpha} = 0,\\
&m_{33} \equiv \alpha\phi_{\alpha,\alpha} + (\alpha + \beta + \gamma)\phi_{3,3} = 0,\\
&m_{3\alpha,\alpha} \equiv \beta\phi_{3,\alpha\alpha} + \gamma\phi_{\alpha,\alpha 3} = 0,\\
&\varepsilon_{\alpha\beta}m_{3\alpha,\beta} \equiv \gamma\varepsilon_{\alpha\beta}\phi_{\alpha,\beta 3} = 0, \qquad \text{on } x_3 = 0
\end{aligned} \tag{16}$$

In these equations and in the equations to follow, Greek indices take values 1, 2 and the symbol $\varepsilon_{\alpha\beta}$ is given by $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$.

3. Surface waves

We consider all possible motions for which p, ξ and ψ have the form

$$\left.\begin{aligned}
p &= f_1(x_3)P_0(x_1, x_2)\exp(i\omega t),\\
\xi &= f_2(x_3)\chi_0(x_1, x_2)\exp(i\omega t),\\
\psi &= f_3(x_3)\Omega_0(x_1, x_2)\exp(i\omega t),
\end{aligned}\right\} \tag{17}$$

where ω is the given frequency, f_i are functions of x_3 tending to zero as $x_3 \to \infty$ and P_0, χ_0 and $\mathbf{\Omega}_0$ are *arbitrary* functions of x_1 and x_2 satisfying the two-dimensional reduced wave equation

$$\left(\frac{\partial^2}{\partial x_\alpha \partial x_\alpha}+\frac{\omega^2}{c^2}\right)F=0. \tag{18}$$

Obviously, the solutions (17) correspond to surface waves of arbitrary form having c as their phase velocity.

The expressions (17) satisfy (6) to (8) and (10), if f_i are given by

$$\left.\begin{aligned} f_1(x_3) &= A_1\exp(-r_1x_3)+A_2\exp(-r_2x_3),\\ f_2(x_3) &= C\exp(-bx_3),\\ f_3(x_3) &= B_1\exp(-s_1x_3)+B_2\exp(-s_2x_3),\end{aligned}\right\} \tag{19}$$

where A_α, B_α and C are arbitrary constants, r_α are roots with positive real parts of the equation

$$(r^2-\sigma^2)^2+(r^2-\sigma^2)[m_1^2-k_3(1+\varepsilon_T\varepsilon_M)]-m_1^2k_3=0, \tag{20}$$

s_α are roots with positive real parts of the equation

$$(s^2-\sigma^2)^2+(s^2-\sigma^2)[m_2^2+m_4^2+\eta(\zeta-2)]+m_2^2(m_4^2-2\eta)=0, \tag{21}$$

and b is the root with positive real part of the equation

$$b^2=\sigma^2+2\nu-m_3^2. \tag{22}$$

In (20) to (22), we have put

$$\sigma=\omega/c,\quad k_3=i\omega/k_2,\quad m_k=\omega/\alpha_k,\quad k=1,2,3,4. \tag{23}$$

With the aid of (19), (17) may be rewritten in the form

$$\left.\begin{aligned} p &= f(x_3)P(x_1,x_2)\exp(i\omega t),\\ \xi &= \chi(x_1,x_2)\exp(-bx_3+i\omega t),\\ \boldsymbol{\psi} &= g(x_3)\boldsymbol{\Omega}(x_1,x_2)\exp(i\omega t),\end{aligned}\right\} \tag{24}$$

where

$$\left.\begin{aligned} f(x_3) &= \exp(-r_1x_3)+A\exp(-r_2x_3),\\ g(x_3) &= \exp(-s_1x_3)+B\exp(-s_2x_3),\end{aligned}\right\} \tag{25}$$

A and B are arbitrary constants and P, χ and $\boldsymbol{\Omega}$ are arbitrary functions of x_α satisfying the wave equation (18).

In view of (6), (8) and (24), $\mathbf{q}$ is of the form

$$\mathbf{q}=h(x_3)\mathbf{Q}(x_1,x_2)\exp(i\omega t), \tag{26}$$

where $h(x_3)$ is completely determined by $g(x_3)$ and $\mathbf{Q}$ by $\boldsymbol{\Omega}$. Like the components of $\boldsymbol{\Omega}$, those of $\mathbf{Q}$ also satisfy (18).

Using (4) to (8), (10), (24) and (26), the boundary conditions (16) yield the following simultaneous equations.

$$(2\sigma^2-m_2^{*2})f(0)P-2h'(0)\varepsilon_{\alpha\beta}Q_{\alpha,\beta}=0, \tag{27a}$$

or

$$(2\sigma^2-m_2^{*2})f(0)P+2M'(0)\Omega_3=0, \tag{27b}$$

$$(2+\varepsilon)f'(0)\sigma^2P-[2\sigma^2-m_2^2+\varepsilon(\sigma^2-m_2^2)]h(0)\varepsilon_{\alpha\beta}Q_{\alpha,\beta}=0, \tag{28a}$$

or $$(2+\varepsilon)f'(0)\sigma^2 P+[2\sigma^2-m_2^2+\varepsilon(\sigma^2-m_2^2)]M(0)\Omega_3=0, \tag{28b}$$

$$(1+\varepsilon)[h''(0)Q_{\alpha,\alpha}+h'(0)\sigma^2 Q_3]-\varepsilon[\sigma^2\chi-g'(0)\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}]=0, \tag{29a}$$

or $$[M'(0)-\zeta g'(0)]\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}+\zeta\sigma^2\chi=0, \tag{29b}$$

and

$$(b^2-m_5\sigma^2)\chi-(1-m_5)g'(0)\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}=0, \tag{30}$$

$$b\sigma^2\chi+[m_6 g''(0)+(1-m_6)\sigma^2 g(0)]\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}=0, \tag{31}$$

$$\sigma^2 g'(0)\Omega_3+g''(0)\Omega_{\alpha,\alpha}=0. \tag{32}$$

In these equations we have put

$$\left.\begin{aligned} &m_2^{*2}=\frac{\omega^2}{c_2^2(1+\varepsilon/2)},\quad c_2^2=\frac{\mu}{\rho},\quad \varepsilon=\frac{\kappa}{\mu},\quad m_5=\frac{\alpha}{\alpha+\beta+\gamma},\quad m_6=\frac{\gamma}{\beta+\gamma}\\ &M(x_3)=\frac{1}{\eta}[(\sigma^2+2\eta-m_4^2)g(x_3)-g''(x_3)]. \end{aligned}\right\} \tag{33}$$

The boundary condition (13) now yields, with the help of (5), (24) and (25), the equation

$$(k-r_1)n_1+A(k-r_2)n_2=0, \tag{34}$$

where

$$n_\alpha=r_\alpha^2+m_1^2-\sigma^2. \tag{35}$$

It is evident that (34) determines the constant A and hence $f(x_3)$ is determined. Further, (30) and (31) yield $B=-N_1/N_2$, where

$$N_\alpha=[(1-m_5)\sigma^2+2\nu-m_3^2][m_6 s_\alpha^2+(1-m_6)\sigma^2]+(1-m_5)b\sigma^2 \tag{36}$$

and consequently the function $g(x_3)$ is determined.

From (27b), (28b), (29b) and (30) to (32), we readily see that the functions P, χ and Ω_3 are all expressible in terms of the functions Ω_1 and Ω_2. Consequently, the functions p, ξ and ψ, which determine all the field variables of the problem are also derivable from the functions Ω_1 and Ω_2. The form of these two functions, which satisfy the reduced wave equation (18), depends on the geometrical form of the wavefront and in the absence of any assumption regarding the nature of the wavefront, these functions are just arbitrary solutions of (18). In this paper we are concerned with waves of arbitrary form and accordingly, in our analysis throughout, the functions Ω_α are regarded as arbitrary functions satisfying just one condition, *viz.* equation (18).

From (27) to (32) we see that P and Ω_3 are linked together and χ is linked only with $\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}$. Accordingly, if one eliminates the unknown functions from these equations, the resulting characteristic equation for the waves splits into two parts. One of these parts [obtained by eliminating P and Ω_3 from (27b) and (28b)] corresponds to a family of coupled waves associated with the displacements, temperature and micro-rotations and the other part [obtained by eliminating χ and $\varepsilon_{\alpha\beta}\Omega_{\alpha,\beta}$ from (29b) and (30)] corresponds to an uncoupled family of waves associated with micro-rotations. These two parts are respectively given by

$$(2\sigma^2 - m_2^{*2})[(2\sigma^2 - m_2^2) + \varepsilon(\sigma^2 - m_2^2)]$$

$$= \frac{2(2+\varepsilon)\sigma^2 M'(0)}{M(0)} \cdot \frac{r_1 r_2(n_1 - n_2) + k(r_1 n_2 - r_2 n_1)}{(n_2 r_2 - n_1 r_1) + k(n_1 - n_2)} \tag{37}$$

$$M'(0)[(1-m_5)\sigma^2 + 2\nu - m_3^2] + \zeta(m_3^2 - 2\nu)g'(0) = 0. \tag{38}$$

If we suppose that the coupling between the micro- and the macro-deformations is so small that terms involving ε^2 may be neglected [10], s_α are given by

$$s_1^2 = \sigma^2[1 - (1-\varepsilon)c^2/c_2^2], \quad s_2^2 = \sigma^2(1 - c^2/\alpha_4^2) + 2\eta, \tag{39}$$

and r_α are governed by the equation

$$(r^2 - \sigma^2)^2 + (r^2 - \sigma^2)[m_1^{*2} - k_3\{1 + \varepsilon_T(1 + \varepsilon\lambda_1)\}] - m_1^{*2}k_3 = 0, \tag{40}$$

where

$$m_1^{*2} = \frac{\omega^2}{c_1^2}(1 - \varepsilon c_3^2),\ c_1^2 = \frac{\lambda + 2\mu}{\rho},\ c_3^2 = \frac{c_2^2}{c_1^2},\ \lambda_1 = \frac{\mu(2\mu - \lambda)}{(\lambda + 2\mu)(3\lambda + 2\mu)}. \tag{41}$$

Equations (37) and (38) now take the form

$$(s_1^2 - s_2^2)\left[(2\sigma^2 - \tau_2^2)^2 - 4\sigma^2(\sigma^2 - \tau_2^2)^{1/2}E + \varepsilon\left\{(2\sigma^2 - \tau_2^2)(\sigma^2 + \tfrac{1}{2}\tau_2^2) - 2\sigma^2(\sigma^2 - \tau_2^2)^{1/2}\left(1 + \frac{\tau_2^2}{\sigma^2 - \tau_2^2}\right)E\right\}\right] = 0, \tag{42}$$

$$(s_1^2 - s_2^2)[(1-m_5)\sigma^2 + 2\nu - m_3^2] = 0, \tag{43}$$

where we have put

$$\tau_2^2 = \frac{\omega^2}{c_2^2} \text{ and } E = E(r_1, r_2) = \frac{(r_1 n_2^* - r_2 n_1^*)k + r_1 r_2(n_1^* - n_2^*)}{(n_2^* - n_1^*)k + (n_1^* r_1 - n_2^* r_2)}, \tag{44}$$

with $$n_\alpha^* = r_\alpha^2 + \sigma^2 - m_1^{*2}. \tag{45}$$

We readily see that the wave motion now takes place in four families. One of these families, governed by the second factor in (42), consists of thermoelastic displacement waves modified under the influence of the micropolar field and the other three families consist of predominantly micropolar waves. Two of these latter families propagate with the *same* frequency given by

$$\omega = c_2(2\rho\varepsilon/I)^{1/2}[1 - \alpha_4^2/c_2^2]^{-1}, \tag{46}$$

which has been obtained from the first factor in each of (42) and (43). Note that these two families disappear as $\varepsilon \to 0$. The third predominantly micropolar family propagates with speed given by

$$c = \left(\frac{\beta + \gamma}{I}\right)^{1/2}\left[1 + \frac{\rho\varepsilon c_2^2}{I\omega^2}\right] \tag{47}$$

which has been obtained from the second factor in (43). Note that this wave is dispersive and in the limiting case when $\varepsilon \to 0$, this propagates with the constant speed $[(\beta + \gamma)/I]^{1/2}$.

It may be mentioned that Suhubi and Eringen [19] (see [10] also) noted the occurrence of *one* family of pure micropolar *plane* surface waves propagating with frequency given by $\omega = c_2(2\rho\varepsilon/I)^{1/2}$. This frequency is clearly an approximation of our expression (46).

The secular equation for the modified thermoelastic waves is given by

$$(2\sigma^2-\tau_2^2)^2-4\sigma^2(\sigma^2-\tau_2^2)^{1/2}E=\varepsilon\Big[(\tau_2^2-2\sigma^2)(\sigma^2+\tfrac{1}{2}\tau_2^2) +2\sigma^2(\sigma^2-\tau_2^2)^{1/2}\Big\{1+\frac{\tau_2^2}{\sigma^2-\tau_2^2}\Big\}E\Big]. \tag{48}$$

As $\varepsilon \to 0$, this equation reduces to

$$(2\sigma^2-\tau_2^2)^2=4\sigma^2(\sigma^2-\tau_2^2)^{1/2}E, \tag{49}$$

where r_α are now governed by the equation

$$(r^2-\sigma^2)^2+(r^2-\sigma^2)[(\omega^2/c_1^2)-k_3(1+\varepsilon_T)]-(\omega^2/c_1^2)k_3=0. \tag{50}$$

Equations (49) and (50) correspond to pure thermoelastic surface waves [2]. Comparison of (48) and (49) shows that the effect of the presence of the micropolar field is to add a correction term to the secular equation of pure thermoelastic surface waves, the correction term being the right side of (48).

If we assume that the frequency is so small that $(\omega/\omega^*) \ll 1$, where $\omega^* = c_1^2/k_2$, the expression (44) yields, for all values of k,

$$E=(\sigma^2-\tau_1^{*2})^{1/2}\Big[1+\tfrac{1}{2}\varepsilon\frac{\tau_1^{*2}(1+c_3^2)-\sigma^2(1-\lambda^*)}{\sigma^2-\tau_1^{*2}}\Big], \tag{51}$$

where we have put

$$\tau_1^{*2}=\omega^2/c_1^2(1+\varepsilon_T),\quad \lambda^*=\lambda_1/(1+\varepsilon_T).$$

The secular equation for the modified thermoelastic waves is then given by

$$(2\sigma^2-\tau_2^2)^2-4\sigma^2(\sigma^2-\tau_2^2)^{1/2}(\sigma^2-\tau_1^{*2})^{1/2}=\varepsilon\Big[(\tau_2^2-2\sigma^2)(\sigma^2+\tfrac{1}{2}\tau_2^2) +2\sigma^2(\sigma^2-\tau_2^2)^{1/2}(\sigma^2-\tau_1^{*2})^{1/2}\Big\{1+\frac{\tau_2^2}{\sigma^2-\tau_2^2}+\frac{\tau_1^{*2}(1+c_3^2)-\sigma^2(1-\lambda^*)}{\sigma^2-\tau_1^{*2}}\Big\}\Big]. \tag{52}$$

If in this equation τ_1^* and λ^* are repalced by $\tau_1 = \omega/c_1$ and λ_1 respectively, the resulting equation is the secular equation for the non-thermal micropolar surface waves. Further, if $\varepsilon \to 0$, we recover the secular equation for pure elastic surface waves obtained in [16].

4. Analysis of deformation

We now obtain $\mathbf{u}$, ϕ and T explicitly in terms of the functions Ω_1 and Ω_2.

Substituting for p from (24)[1] and for curl $\mathbf{q}$ from (8) in (4) and using (27b) and (32), we obtain the following expressions for the components of $\mathbf{u}$.

$$\left.\begin{aligned} u_\alpha &= [F(x_3)\Omega_{\beta,\beta\alpha} + M(x_3)\Omega_\alpha]\exp(i\omega t), \\ \text{and} \quad u_3 &= G(x_3)\Omega_{\beta,\beta}\exp(i\omega t), \end{aligned}\right\} \tag{53}$$

where

$$\left.\begin{aligned} F(x_3) &= \frac{2M'(0)g''(0)f(x_3)}{\sigma^2 f(0)g'(0)(2\sigma^2 - m_2^{*2})}, \\ \text{and} \quad G(x_3) &= F'(x_3) - \frac{g''(0)}{\sigma^2 g'(0)} M(x_3). \end{aligned}\right\} \tag{54}$$

If we compute curl ψ from (24) on using (18) and (31) and put this into $(4)^2$, we obtain, on using (29), the following expressions for the components of ϕ.

$$\left.\begin{aligned} \phi_\alpha &= [L(x_3)\varepsilon_{\beta\gamma}\Omega_{\beta,\gamma\alpha} + N(x_3)\varepsilon_{\alpha\beta}\Omega_\beta]\exp(i\omega t), \\ \text{and} \quad \phi_3 &= R(x_3)\varepsilon_{\beta\gamma}\Omega_{\beta,\gamma}\exp(i\omega t), \end{aligned}\right\} \tag{55}$$

where

$$\left.\begin{aligned} L(x_3) &= \frac{g'(0)(1-m_5)}{b^2 - m_5\sigma^2}\exp(-bx_3) - \frac{g''(0)}{\sigma^2 g'(0)} g(x_3), \\ N(x_3) &= \frac{g''(0)}{g'(0)} g(x_3) - g'(x_3), \\ R(x_3) &= -\left[g(x_3) + \frac{bg'(0)(1-m_5)}{b^2 - m_5\sigma^2}\exp(-bx_3)\right]. \end{aligned}\right\} \tag{56}$$

Equations (5), $(24)^1$, (26) and (31) yield

$$T = S(x_3)\Omega_{\beta,\beta}\exp(i\omega t), \tag{57}$$

where

$$S(x_3) = \frac{2(\lambda+2\mu+\kappa)M'(0)g''(0)[(m_1^2-\sigma^2)f(x_3)+f''(x_3)]}{(3\lambda+2\mu+\kappa)af(0)g'(0)(2\sigma^2 - m_2^{*2})\sigma^2}. \tag{58}$$

Expressions (53) to (58) thus determine all the field variables in terms of Ω_1 and Ω_2. We now introduce four new functions Γ_α and θ_α of x_α through the relations

$$\Omega_1 = \Gamma_1\exp(\theta_1), \quad \Omega_2 = \Gamma_2\exp(\theta_2). \tag{59}$$

Putting these into (53) and suppressing $\exp(i\omega t)$, we get

$$\left.\begin{aligned} u_1 &= [H_{11}F + \Gamma_1 M]\exp(\theta_1) + H_{21}F\exp(\theta_2), \\ u_2 &= H_{12}F\exp(\theta_1) + [H_{22}F + \Gamma_2 M]\exp(\theta_2), \\ u_3 &= G[\kappa_{11}\exp(\theta_1) + \kappa_{22}\exp(\theta_2)], \end{aligned}\right\} \tag{60}$$

where we have put

$$\kappa_{\alpha\beta} = \Gamma_{\alpha,\beta} + \Gamma_\alpha\theta_{\alpha,\beta}, \quad H_{\alpha\beta} = \kappa_{\alpha\alpha,\beta} + \kappa_{\alpha\alpha}\theta_{\alpha,\beta} \text{ (no summation!)} \tag{61}$$

Eliminating θ_α from (60) we obtain the following linear equation connecting the components of **u**.

$$[(\kappa_{11}H_{22}-\kappa_{22}H_{12})F+\kappa_{11}M\Gamma_2]Gu_1$$
$$+[(\kappa_{22}H_{11}-\kappa_{11}H_{21})F+\kappa_{22}M\Gamma_1]Gu_2$$
$$+[(H_{21}H_{12}-H_{11}H_{22})F^2-(H_{11}\Gamma_2+H_{22}\Gamma_1)MF-\Gamma_1\Gamma_2M^2]u_3=0. \tag{62}$$

Clearly, at any point of the body u_1, u_2, u_3 form a plane and the orientation of this plane relative to the boundary $x_3 = 0$ changes from point to point, because of the dependence on x_i of the coefficients of u_i in (62). This plane becomes perpendicular to the boundary if and only if the coefficient of u_3 in (62) is zero. Accordingly, the elastic deformation is *not* in general confined to a plane perpendicular to the boundary.

Introducing (59) into (55) and suppressing $\exp(i\omega t)$, we get

$$\left.\begin{aligned}\phi_1&=(\kappa_{12,1}+\kappa_{12}\theta_{1,1})L\exp(\theta_1)-[(\kappa_{21,1}+\kappa_{21}\theta_{2,1})L-\Gamma_2N\}\exp(\theta_2)\\ \phi_2&=[(\kappa_{12,2}+\kappa_{12}\theta_{1,2})L-\Gamma_1N]\exp(\theta_1)-(\kappa_{21,2}+\kappa_{21}\theta_{2,2})L\exp(\theta_2)\\ \phi_3&=R[\kappa_{12}\exp(\theta_1)-\kappa_{21}\exp(\theta_2)].\end{aligned}\right\} \tag{63}$$

Eliminating θ_α from these equations we obtain the following linear equation connecting the components of $\boldsymbol{\phi}$.

$$[\kappa_{21}(LD_1-N\Gamma_1)-\kappa_{12}LD_2]R\phi_1+[\kappa_{12}(LC_2-N\Gamma_2)-\kappa_{21}LC_1]R\phi_2$$
$$+[C_1D_2L^2-(LC_2-N\Gamma_2)(LD_1-N\Gamma_1)]\phi_3=0, \tag{64}$$

where we have put

$$\left.\begin{aligned}C_1&=\kappa_{12,1}+\kappa_{12}\theta_{1,1};\quad C_2=\kappa_{21,1}+\kappa_{21}\theta_{2,1},\\ D_1&=\kappa_{12,2}+\kappa_{12}\theta_{1,2};\quad D_2=\kappa_{21,2}+\kappa_{21}\theta_{2,2}.\end{aligned}\right\} \tag{65}$$

We see that the microrotation at a point also takes place in a plane and that the orientation of this plane relative to the boundary $x_3 = 0$ also varies from point to point in the body. Note that the angle between the planes (64) and (62) also varies from point to point.

In the *particular case* of plane waves propagating in a fixed direction parallel to the boundary, say x_1-direction, the functions Ω_α are independent of the x_2-coordinate and (60) and (62) to (64) then reduce to the following form.

$$u_1=(H_{11}F+\Gamma_1M)\exp(\theta_1)+H_{21}F\exp(\theta_2),$$
$$u_2=M\Omega_2,\quad u_3=\kappa_{11}G\exp(\theta_1), \tag{66}$$
$$\kappa_{11}G(M\Gamma_2u_1-H_{21}Fu_2)-M\Gamma_2(H_{11}F+M\Gamma_1)u_3=0, \tag{67}$$
$$\phi_1=(N\Gamma_2-LC_2)\exp(\theta_2),\quad \phi_2=-N\Omega_1,\quad \phi_3=-R\kappa_{21}\exp(\theta_2), \tag{68}$$
$$R\kappa_{21}\phi_1+(N\Gamma_2-LC_2)\phi_3=0. \tag{69}$$

From (66) and (67) we note that, unlike in the corresponding situation in classical elasticity and thermoelasticity, u_2 is *not* necessarily zero and the motion is therefore not necessarily confined to the x_2- plane. From (68) and (69) we note that the microrotation at every point takes place in a plane parallel to the x_2-plane and that no component of $\boldsymbol{\phi}$ is zero, in general.

In [19], Suhubi and Eringen studied non-thermal micropolar plane surface waves on

the hypothesis that the displacement vector lies completely in the plane determined by the direction of propagation and the normal to the boundary and the microrotation vector always lies perpendicular to this plane. Obviously this hypothesis corresponds to the special case in which $\Omega_2 \equiv 0$. The single scalar function Ω_1 then determines all the field variables associated with the waves.

In the *limiting case* when $\varepsilon \to 0$, $\mathbf{q}$ and ψ and consequently $g(x_3)$ and $h(x_3)$ become uncoupled. Equations (6) and (8) then yield

$$\left.\begin{aligned} &h(x_3) = \exp(-s_1 x_3), \quad g(x_3) = \exp(-s_2 x_3), \\ \text{where} \quad &s_1^2 = \sigma^2\left(1 - \frac{c^2}{c_2^2}\right), \qquad s_2^2 = \sigma^2\left(1 - \frac{c^2}{\alpha_4^2}\right), \end{aligned}\right\} \tag{70}$$

and (22) gives

$$b^2 = \sigma^2 - m_3^2. \tag{71}$$

Putting (70) and (71) into (55) and (56), we get

$$\left.\begin{aligned} &\phi_\alpha = \frac{s_2}{\sigma^2}\left[\exp(-s_2 x_3) - \frac{(1-m_5)\exp(-bx_3)}{(1-m_5) - (c^2/\alpha_3^2)}\right]\varepsilon_{\beta\gamma}\Omega_{\beta,\gamma\alpha}\exp(i\omega t), \\ \text{and} \quad &\phi_3 = \left[\frac{bs_2(1-m_5)\exp(-bx_3)}{(1-m_5)\sigma^2 - (c^2/\alpha_3^2)} - \exp(-s_2 x_3)\right]\varepsilon_{\beta\gamma}\Omega_{\beta,\gamma}\exp(i\omega t). \end{aligned}\right\} \tag{72}$$

Clearly, the micropolar field is now completely determined by the *single* scalar function $\varepsilon_{\beta\gamma}\Omega_{\beta,\gamma}$.

Putting the expression for $h(x_3)$ from (70) into (29a) (with $\varepsilon \to 0$), we get

$$\sigma^2 Q_3 = s_1 Q_{\alpha,\alpha}. \tag{73}$$

The vector $\mathbf{Q}$ is thus completely determined by its first two components. Equations $(4)^1$, $(24)^1$, (26), (27a) and (73) now yield the following expressions for the components of $\mathbf{u}$ and T:

$$\left.\begin{aligned} u_\alpha &= \left[f(x_3) - f(0)\left(1 - \frac{c^2}{2c_2^2}\right)\exp(-s_1 x_3)\right]P_{,\alpha}\exp(i\omega t), \\ u_3 &= \left[f'(x_3) + \frac{f(0)}{s_1}\left(\sigma^2 - \frac{\omega^2}{2c_2^2}\right)\exp(-s_1 x_3)\right]P\exp(i\omega t), \\ T &= \frac{\lambda + 2\mu}{(3\lambda + 2\mu)a}\left[\left(\frac{\omega^2}{c_1^2} - \sigma^2\right)f(x_3) + f''(x_3)\right]P\exp(i\omega t). \end{aligned}\right\} \tag{74}$$

Clearly, the thermoelastic field is now determined by the *single* scalar function P. The expressions (74) are in agreement with the corresponding expressions obtained in [2].

Finally, it may be remarked that one may generalize (in a rather trivial way) the problem considered in this paper by employing the equations of generalized thermoelasticity theory [4, 5] in place of those of the coupled thermoelasticity theory (employed here). For example, our entire analysis holds in the context of the Lord–Shulman theory [17], if (20) is replaced by

$$\begin{aligned} &(r^2 - \sigma^2)^2 + (r^2 - \sigma^2)[m_1^2 - k_3(1 + i\omega k_4)(1 + \varepsilon_T\varepsilon_M)] \\ &- m_1^2 k_3(1 + i\omega k_4) = 0. \end{aligned} \tag{75}$$

Similarly, if the equations obtained in [8] (which are based on the Green–Lindsay theory [14]) are employed, (20) is to be replaced by the equation

$$(r^2-\sigma^2)^2+(r^2-\sigma^2)[m_1^2-k_3\{(1+i\omega k_6)+(1+i\omega k_5)\varepsilon_T\varepsilon_M\}]$$
$$-m_1^2k_3(1+i\omega k_6)=0. \tag{76}$$

In (75) and (76), k_4, k_5 and k_6 are (thermal) constants satisfying the conditions $k_4 \geqslant 0$, $k_5 \geqslant k_6 \geqslant 0$.

In [1], Agarwal studied non-polar thermoelastic surface waves in the context of the Lord–Shulman and the Green–Lindsay theories. His analysis follows as a special case our analysis if we set $\phi = 0$ in our analysis and take $\varepsilon_M = 1$ in (75) and (76) above. Indeed, when r_1 and r_2 are governed by (75) with $\varepsilon_M = 1$, our equations (74) correspond to the equations (2.27) of [1].

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On the uniform Nörlund summability of Fourier series

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Abstract. A general theorem on uniform Nörlund summability of Fourier series has been derived. Some known results become its special cases.

Keywords. Nörlund summability; Fourier series

1. Introduction

Let $f(x)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. The Fourier series associated with $f(x)$ is given by

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx). \tag{1}$$

Write

$$\varphi(t) = \tfrac{1}{2}[f(x+t)+f(x-t)-2f(x)],$$
$$\Phi(t) = \int_0^t |\varphi(u)|\,du.$$

Concerning Nörlund summability (N, p_n), many workers have obtained interesting results for Fourier series. In this paper we give the concept of uniform Nörlund summability which is more general than ordinary Nörlund summability. Our theorem in this direction for Fourier series includes a number of known results as its particular cases.

2. Uniform Nörlund summability

DEFINITION:

Let $\{p_n\}$ be a sequence of real numbers such that

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \text{ (for } n \geqslant 0),$$

and let E be a set of real numbers. The series $\Sigma u_n(x), x \in E$ is said to be summable (N, p_n) uniformly to $s(x)$ in E as $n \to \infty$ if $t_n(x)$ converges uniformly to $s(x)$ in E when $n \to \infty$, where,

$$t_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k(x). \tag{2}$$

From the above definition it is obvious that when E contains only one point the above definition reduces to ordinary Nörlund summability. In fact in the uniform Nörlund summability we study the behaviour of $\{t_n(x)\}$ over a set E instead of a particular point.

3. The sequence $\{p_n\}$

Define $p(t)$ on $(0, \infty)$ such that $p_n = p(n)$, $n = 0, 1, \ldots$ and that $p(t)$ is continuous on $(0, \infty)$ and linear in each $(K, K+1)$, $K = 0, 1, \ldots$.

Put

$$P(t) = \int_0^t p(u)\,du$$

then

$$P(n) = \tfrac{1}{2}p_0 + \ldots + p_{n-1} + \tfrac{1}{2}p_n \simeq P_n$$

as $n \to \infty$, when $P_n \to \infty$ and $p_n/P_n \to 0$. Let $p'(u)$ denote the derivative of $p(u)$. We observe that

$$\int_1^n u|p'(u)|\,du = O(P_n) \tag{3}$$

as $n \to \infty$, is more general than $\{p_n\}$, non-negative and non-increasing. If $\{p_n\}$ is monotonic (3) is equivalent to $np_n = O(P_n)$ (This can be verified by integrating in parts the left side of (3)). Again, if $\{p_n\}$ is decreasing, then $np_n = O(P_n)$ is satisfied. Hence, in general, (3) implies non-negative and non-increasing nature of the sequence of coefficients $\{p_n\}$.

4. The main result

We establish the following:

THEOREM:

Let $\{p_n\}$ be a positive sequence such that $P_n \to \infty$ as $n \to \infty$ and

$$\int_1^n u|p'(u)|\,du = O(P_n) \tag{4}$$

as $n \to \infty$. Suppose $\varepsilon(x)$ be a non-negative function of x such that $\varepsilon(x)/x \log x$ is monotonic, $\varepsilon(x) \log x = O(1)$ as $x \to \infty$ and

$$\sum_{k=2}^{n} \frac{\varepsilon(k)P_k}{k \log k} = O(P_n), \tag{5}$$

as $n \to \infty$. Then, if

$$\Phi(t) = o\left[\frac{t\varepsilon(1/t)}{\log 1/t}\right], \tag{6}$$

as $t \to 0$ uniformly in the set E then the Fourier series of $f(x)$ is summable (N, p_n) uniformly in E to $f(x)$.

5. Proof of the theorem

The nth partial sum $S_n(x)$ of the Fourier series (1) is given by

$$S_n(x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\varphi(t)\,dt.$$

Hence

$$\frac{1}{P_n}\sum_{k=0}^{n}p_k\{S_{n-k}(x)-f(x)\}=O[\int_0^{\pi}K(n,t)\varphi(t)dt];$$

where

$$K(n,t)=\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{1}{2}t}.$$

Now,

$$O[\int_0^{\pi}K(n,t)\varphi(t)dt]=O[\int_0^{n^{-1}}+\int_{n^{-1}}^{\pi}]K(n,t)\varphi(t)dt=I+J. \tag{7}$$

In order to prove the theorem we have to show that $I+J=o(1)$ as $n\to\infty$ uniformly in the set E.

Since $K(n,t)=O(n)$, $O<t<n^{-1}$.
Hence

$$I=O[n\int_0^{n^{-1}}|\varphi(t)|\,dt]=o\left[\frac{\varepsilon(n)}{\log n}\right]=o(1),$$

as $n\to\infty$ uniformly in E, under the condition (6) and using the hypothesis of the theorem. Write

$$\begin{aligned}|L_n(t)|&=\left|\sum_{k=0}^{n}p_k\sin(n-k+\tfrac{1}{2})t\right|\\&\leqslant\sum_{k=0}^{\tau}p_k+\left|\sum_{k=\tau}^{n}p_k\sin(n-k+\tfrac{1}{2})t\right|\\&=L_n'(t)+L_n''(t),\end{aligned} \tag{8}$$

where $\tau=[1/t]$. By Abel's transformation we get

$$|L_n''(t)|\leqslant A\left\{\frac{p_\tau}{t}+\frac{p(n)}{t}+\frac{1}{t}\int_{1/t}^{n}|p'(u)|\,du\right\}. \tag{9}$$

Now,

$$\begin{aligned}|J|&=O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi}\frac{\varphi(t)}{\sin\frac{1}{2}t}L_n(t)dt\right]\\&=O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi}(L_n'(t)+L_n''(t))\frac{\varphi(t)}{\sin\frac{1}{2}t}dt\right]\end{aligned}$$

in view of (8)

$$= J_1 + J_2$$

$$|J_1| = O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} \frac{|\varphi(t)|}{t} P_\tau \, dt\right]$$

$$= O\left[\frac{1}{P_n}\left\{\left[\Phi(t)\frac{P_\tau}{t}\right]_{n^{-1}}^{\pi} + \int_{n^{-1}}^{\pi} \frac{\Phi(t)}{t^2} P_\tau \, dt + \int_{n^{-1}}^{\pi} \frac{\Phi(t)}{t} dP_\tau\right\}\right]$$

$$= J_{1.1} + J_{1.2} + J_{1.3},$$

$$J_{1.1} = O\left[\frac{1}{P_n}\left\{o\left(\frac{t\varepsilon(1/t)P_\tau}{t\log\frac{1}{t}}\right)_{n^{-1}}^{\pi}\right\}\right] \text{ by (6)}$$

$$= o\left[\frac{1}{P_n}\frac{\varepsilon(n)}{\log n} P_n\right] = o(1)$$

as $n \to \infty$ uniformly in E under the hypothesis of the theorem. Again

$$J_{1.2} = o\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} \frac{t\varepsilon(1/t)}{\log\frac{1}{t}t^2} P_\tau \, dt\right] \text{ by (6)}$$

$$= o\left[\frac{1}{P_n}\int_{n}^{1/\pi} \frac{\varepsilon(x)}{x\log x} P_{[x]} \, dx\right]$$

$$= o\left[\frac{1}{P_n}\sum_{x=2}^{n} \frac{\varepsilon(x)P(x)}{x\log x}\right] \text{ by (5)}$$

$$= o(1) \text{ as } n \to \infty, \text{ uniformly in } E$$

and

$$J_{1.3} = O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} o\left\{\frac{t\varepsilon(1/t)}{t\log(1/t)}\right\} dP_\tau\right] \text{ by (6)}$$

$$= o\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} dP_\tau\right]$$

$$= o(1), \text{ as } n \to \infty, \text{ uniformly in } E.$$

In view of (9) we have

$$J_2 = O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} \frac{|\varphi(t)|}{t}\frac{p_\tau}{t} dt + \int_{n^{-1}}^{\pi} \frac{|\varphi(t)|}{t}\frac{p(n)}{t} dt + \int_{n^{-1}}^{\pi} \frac{|\varphi(t)|}{t^2} dt \int_{1/t}^{n} |p'(u)| \, du\right]$$

$$= O\left[\frac{1}{P_n}\int_{n^{-1}}^{\pi} \frac{|\varphi(t)|}{t} P_\tau \, dt + p(n)\left(\frac{\Phi(t)}{t^2}\right)_{n^{-1}}^{\pi} + p(n)\int_{n^{-1}}^{\pi} \frac{\Phi(t)}{t^3} dt \int_{1/t}^{n} |p'(u)| \, du + \int_{n^{-1}}^{\pi} \frac{\Phi(t)}{t^2}\frac{|p'_\tau|}{t^2} dt\right]$$

$$= O(J_1) + \frac{1}{P_n} o(n p_n) + o\left[\frac{1}{P_n}\int_1^n |p'(u)|\,du\right]$$
$$+ o\left[\frac{1}{P_n}\int_1^n u|p'(u)|\,du\right] + o\left[\frac{1}{P_n}\int_1^n u|p'(u)|\,du\right]$$
$$= o(1),$$

as $n \to \infty$ uniformly in E, $J_1 = o(1)$ as $n \to \infty$ uniformly in E as shown above. Thus we get $I + J = o(1)$ as $n \to \infty$, uniformly in E, which establishes the truth of (7).

Special Cases

Taking $p_n = 1/(n+1)$, $P_n \sim \log n$ as $n \to \infty$, and $\varepsilon(x) = 1$ we note that the known results due to Hille and Tamarkin [1], Iyengar [3] and Prasad and Siddiqi [5] become special cases of our theorem.

With $\varepsilon(x) = 1$ our theorem includes the known result of Singh [6] under minimum restriction on $\{p_n\}$ and more general summability method. The theorem also extends various other known results, *e.g.* Pat [4] and Hsiang [2] etc. under general summability method and minimum restriction on $\{p_n\}$.

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Solvability of generalized Hammerstein equations

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Abstract. In this paper we obtain solvability results for generalized Hammerstein equations by using the theory of k-set contractions.

Keywords. Generalized Hammerstein equations; k-set contractions; Chandrasekhar's H-equation.

1. Introduction

Let X and Y be Banach spaces. In this paper we obtain existence results for generalized Hammerstein equations of the form

$$x = x_0 + K(x)Nx, \tag{1}$$

where for each $x \in X$, $K(x)$: $Y \to X$ is a bounded linear operator and $N: X \to Y$ is a bounded nonlinear operator.

The motivation for the investigation of solvability of (1) arose from the study of nonlinear integral equations of the form

$$x(s) = x_0(s) + \int_\Omega K(x; s, t) f[t, x(t)]\,dt, \tag{2}$$

where Ω is a bounded subset of R^n and kernel $K(x; s, t)$ is the function of the solution x. If $K(x; s, t) = K(s, t)$ for all x, we obtain the usual Hammerstein integral equation. Equations of the type (2) occur as integral analogue of some quasilinear boundary value problems (refer §3). Also they occur independently in many physical problems [9, 14, 15] and contain the well-known Chandrasekhar's H-equation [4]

$$x(s) = 1 + x(s) \int_0^1 \left(\frac{s}{s+t}\right) \psi(t) x(t)\,dt \tag{3}$$

as a special case.

The generalized Hammerstein equations have been studied by Avramescu [1], Petry [10, 11, 12] and more recently by Backwinkel-Schillings [2, 3], Srikanth and Joshi [13] and Joshi [7]; the theory of monotone operators has been effectively used for such equations to obtain solvability results. The main aim of this paper is to use the theory of strict set contractions to get rid of monotonicity type conditions on $K(x)$ and N at the expense of Lipschitz continuity on N, which is at times helpful. Our approach is influenced by the main result of Legget [8].

2. Abstract theorems

DEFINITION 2.1

Let M be a metric space and let A be a bounded subset of M. $r(A)$, the measure of noncompactness is defined as $r(A) = \inf\{d > 0 : A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$. One can show that a bounded subset A of a complete metric space is relatively compact iff $r(A) = 0$.

DEFINITION 2.2

Let M and N be two metric spaces, then $T: M \to N$ is said to be a k-set contraction if there exists $k \geqslant 0$ such that $r[T(A)] \leqslant k\, r(A)$ for all bounded subsets A of M. If $k < 1$ in the above inequality then T is said to be a strict set contraction.

In the following we derive conditions under which the map $Tx = x_0 + K(x)Nx$ is a k-set contraction.

THEOREM 2.1

Let X and Y be two Banach space and A a subset of X. Let for each $x \in A$, $K(x): Y \to X$ be a bounded linear operator and $N: A \to Y$ a bounded nonlinear operator. Further, assume that the following hold:

(a) $x \to K(x)$ is compact as a map from A to $L(Y, X)$;
(b) N is Lipschitz continuous with constant α, that is,

$$\|Nx - Ny\| \leqslant \alpha \|x - y\| \text{ for all } x, y \in A;$$

(c) $\beta = \sup_{x \in A} \|K(x)\| < \infty$.

If $\alpha\beta < 1$, then T is a strict set contraction.

Proof. Since the measure of noncompactness r is translation invariant, it suffices to consider the case $x_0 = 0$.

Let C be a bounded subset of A and let $\varepsilon > 0$ be arbitrary. Assume that $x^* \in C$ is a fixed element and $\delta = \operatorname{diam} N(C)$. Choose sets $C_1, C_2, \ldots, C_n$ such that $\operatorname{diam} C_i \leqslant r(C) + \varepsilon$ and $C = \bigcup_{i=1}^{n} C_i$. Since $x \to K(x)$ is compact as a map from A to $L(Y, X)$, there exist finitely many set $G_1, G_2, \ldots, G_m$ in $L(Y, X)$ such that $\operatorname{diam} G_i < [\|Nx^*\|]^{-1}\varepsilon$, $i = 1, \ldots m$ and $K(C) = \bigcup_{i=1}^{m} G_i$, where $K(C)$ is defined as $K(C) = [K(x): x \in C] \subset L(Y, X)$. Define sets $B_{i,j}$, $i = 1, \ldots n$; $j = 1, \ldots m$ by

$$B_{i,j} = \{K(x)Nx : x \in C_i,\ K(x) \in G_j\}.$$

It easily follows that $\operatorname{diam} B_{i,j} \leqslant \alpha\beta r(C) + \varepsilon$ and hence $r(T(C)) \leqslant \alpha\beta r(C)$. Since $\alpha\beta < 1$, it follows that T is a strict set contraction.

This gives us a solvability result for generalized Hammerstein equation

$$x = x_0 + K(x)Nx \tag{4}$$

in a Banach space X.

In the following, A denotes the closed ball $\|x - x_0\| \leqslant \rho$ in X, where ρ is to be defined later.

THEOREM 2.2

Let X and Y be Banach spaces and $x_0 \in X$. Let for each $x \in A$, $K(x): Y \to X$ be a bounded linear operator and $N: A \to Y$ a bounded nonlinear operator satisfying assumptions (a), (b), (c) of Theorem 2.1. Let α, β, ρ be constants such that $\alpha\beta < 1$ and $\rho \geqslant [\beta\|Nx_0\|/(1-\alpha\beta)]$. Then (4) has solution in the ball A.

Proof: Let $Tx = x_0 + K(x)Nx$. Then T maps A into itself, for

$$\begin{aligned}\|Tx - x_0\| &= \|K(x)Nx\| \\ &\leqslant \|K(x)\|[\|Nx - Nx_0\| + \|Nx_0\|] \\ &\leqslant \beta[\alpha\|x - x_0\| + \|Nx_0\|] \\ &\leqslant \beta[\alpha\rho + \|Nx_0\|] \\ &\leqslant \rho.\end{aligned}$$

Under assumptions (a), (b), (c) of Theorem 2.1, T is a strict set contraction. Thus T is a strict set contraction mapping a closed, convex, bounded subset A of a Banach space X into itself and hence it follows by a theorem of Darbo [5] that T has a fixed point in A. This proves the theorem.

We now obtain a theorem giving a monotonic sequence converging to a solution of (4).

DEFINITION 2.3

Let X be a Banach space and C a subset of X. Then C is called a cone if it satisfies the following:
(a) C is closed in X;
(b) whenever $x, y \in C$ then $\alpha x + \beta y \in C$ for $\alpha, \beta \geqslant 0$;
(c) if $x \neq 0 \in C$ then $-x \notin C$.
Cone C allows a partial ordering in the space X by defining $x \leqslant y$ iff $y - x \in C$.

DEFINITION 2.4

Let X be a Banach space-partially ordered by a cone C. Then norm in X is called monotonic if $0 \leqslant x \leqslant y$ implies that $\|x\| \leqslant \|y\|$.

It is easy to see that the norm in spaces $C[0, 1]$ and $L_p[0, 1]$ is monotonic with respect to the partial ordering generated by the cone of non-negative functions.

DEFINITION 2.5

Let X and Y be partially ordered Banach spaces. $T: X \to Y$ is called isotone if $x \leqslant y$ implies that $Tx \leqslant Ty$.

Let $f(t, x): [0, 1] \times R \to R$ satisfy the Caratheodory condition, that is, $f(t, x)$ is measurable in t for all $x \in R$ and continuous in x for almost all $t \in [0, 1]$. Further assume that f is monotonically increasing with respect to x for all t and satisfies a

growth condition of the type:

$$|f(t, x)| \leqslant a(t) + b|x|^{p/q} (1/p + 1/q = 1),\ a \in L_q,\ b > 0.$$

Then the operator N defined as $[Nx](t) = f[t, x(t)]$ is isotone as a map from $L_p[0, 1]$ to $L_q[0, 1]$.

THEOREM 2.3

Let X and Y be Banach spaces partially ordered by cones C and D respectively and let the norm in X be monotonic. Let for each $x \in C$, $K(x)$ be a bounded linear operator from Y to X mapping D into C and N a bounded nonlinear operator from X to Y mapping C into D. Further, assume that the following hold:
(a) $x \to K(x)$ is compact as a map from C to $L(Y, X)$;
(b) $x_1 \leqslant x_2$ in C implies that $K(x_1)y \leqslant K(x_2)y$ for all $y \in D$;
(c) N is isotone and Lipschitz continuous with constant α on C.
Let a sequence $\{x_n\}$ in C be defined by

$$x_{n+1} = x_0 + K(x_n)Nx_n,\ x_0 \in C.$$

If $\alpha\beta < 1 (\beta = \sup \|K(x_n)\|)$, then the sequence $\{x_n\}$ converges to a solution x of (4) with

$$\|x\| \leqslant \frac{\|x_0\| + \beta\|N(0)\|}{(1 - \alpha\beta)}.$$

Proof: Let T on C be defined by $Tx = x_0 + K(x)Nx$. Then $x_{n+1} = Tx_{n+1}$, $n = 0, 1, \ldots$ and so it maps the set $A = \{x_0, x_1, \ldots.\}$ into itself. We claim that $\{x_n\}$ is a monotonically increasing sequence. This is proved by induction.

We already have $x_0 \leqslant x_1$ and assume that $x_{n-1} \leqslant x_n$. Then

$$\begin{aligned} x_{n+1} - x_n &= K(x_n)Nx_n - K(x_{n-1})Nx_{n-1} \\ &= K(x_n)[Nx_n - Nx_{n-1}] + [K(x_n) - K(x_{n-1})]Nx_{n-1}. \end{aligned}$$

In view of the isotonicity of N and the assumption (b), both terms on the right side of the above equality belong to C and hence $x_n \leqslant x_{n+1}$. This proves the claim.

By monotonicity of norm, it now follows that

$$\|x_n\| \leqslant \|x_{n+1}\| \leqslant \|x_0\| + \alpha\beta\|x_n\| + \beta\|N(0)\|, \text{ for all } n.$$

This gives

$$\|x_n\| \leqslant \frac{\|x_0\| + \beta\|N(0)\|}{(1 - \alpha\beta)} \text{ for all } n.$$

So A is a bounded set with $A = \{x_0\} \cup T(A)$, which implies that $r(A) = r[T(A)]$. Since T is a strict set contraction, it follows that $r = 0$ and hence A is relatively compact. So there exists a subsequence of $\{x_n\}$ which converges to some x and monotonicity of the sequence implies that the entire sequence $\{x_n\}$ converges to x. Continuity of T gives $Tx = x$ and hence the result.

3. Applications

We first consider the following quasilinear boundary value problem introduced by Backwinkel-Schillings in [2, 3]:

$$\left\{\left[\left(\frac{1}{t}\right)+a(t,x(t))\right]^{-1}x'(t)\right\}'+\left\{\left(\frac{1}{t}\right)+a[t,x(t)]\right\}x(t)=f[t,x(t)] \tag{5}$$

$$x(0) \text{ is bounded; } x(1)+x'(1)=0. \tag{5a}$$

Assume the following hypotheses on f and a

[A] (1) $f:[0,1]X\,R\to R$ satisfies the Caratheodory condition.

(2) f is bounded on bounded subsets of $[0,1]X\,R$.

(3) There exist constants α and ρ with $0<\alpha<2$ and $\rho>0$ such that

$$|f(t,x_1)-f(t,x_2)|\leqslant\alpha|x_1-x_2| \text{ for } x_1,x_2\leqslant\rho \text{ and all } t\in[0,1].$$

[B] $a:[0,1]X\,R\to R$ is continuous and for each $x\in R$, $a(1,x)=0$.

DEFINITION 3.1

A function x is said to be a solution of the boundary value problem (5) if x has a continuous derivative in (0, 1), satisfies the boundary condition (5a) and satisfies the differential equation (5) almost everywhere in (0, 1).

THEOREM 3.1

Under assumptions [A] and [B] on f and a respectively, there exists a solution $x\in L_\infty(0,1)$ of the boundary value problem (5) satisfying $\|x\|_{L_\infty}\leqslant\rho$, provided

$$\rho>\frac{\|N(0)\|_{L_1}}{(z-\alpha)} \text{ where } N(0) \text{ is the function } f(t,0).$$

Proof: Solvability of the boundary value problem (5) is equivalent to the solvability of the generalized Hammerstein equation

$$x(s)=\int_0^1 K(x;s,t)f[t,x(t)]\,dt, \tag{6}$$

where $K(x;s,t)$ is Green's function of the differential operator $L(x)$: $[L(x)y](t)=\{[(1/t)+a(t,x(t))]^{-1}y'(t)\}'+\{(1/t)+a[t,x(t)]\}y(t)$. (refer Backwinkel-Schillings [3], pp 192–193).

Now set $X=L_\infty(0,1)$, $Y=L_1(0,1)$ and define $K(x)$: $Y\to X$ and $N:X\to Y$ as follows

$$[K(x)y](s)=\int_0^1 K(x;s,t)\,y(t)\,dt \tag{7}$$

$$[Nx](s)=f[s,x(s)]. \tag{7a}$$

In operator theoretic terms (6) is equivalent to

$$x=K(x)Nx. \tag{8}$$

In view of assumption [A], N is a bounded nonlinear operator from X to Y, satisfying Lipschitz condition

$$\|Nx_1-Nx_2\|\leqslant\alpha\|x_1-x_2\|, \text{ for } x_1,x_2\in A=\{x:\|x\|\leqslant\rho\}.$$

Also one can show that $x\to K(x)$ is compact as a mapping from A to $L(Y,X)$ with $\|K(x)\|\leqslant 1/2$ for all x (refer Backwinkel-Schillings [2]). Thus $K(x)$ and N satisfy all conditions of Theorem 2.2 and hence (8) has a solution $x\in X$ with $\|x\|\leqslant\rho$. This, in turn, implies that (6) and hence (5) has a solution $x\in L_\infty(0,1)$.

Remark 1. The main theorem of Legget [8] cannot be used to obtain solvability-

result for (5) because the mapping $K(x)$ is not compact as a linear operator from $L_1(0, 1)$ to $L_\infty(0, 1)$. Moreover, the domain space of $K(x)$ is not a Banach algebra and hence the result of Legget is not much of a help in this particular example. To this extent our result could be considered more general.

Remark 2. Our result for the boundary value problem (5) is different from that of Backwinkel-Schillings [3] in the sense that we assume Lipschitz continuity of f whereas in [3] monotonicity type condition is assumed.

We now discuss the solvability of generalized Hammerstein integral equation

$$x(s) = x_0(s) + \int_0^1 K(x; s, t) f[t, x(t)]\, dt \tag{9}$$

in the cone C of non-negative functions.

ASSUMPTIONS [C]

(i) $f: [0, 1] X R \to R$ satisfies Caratheodory condition and a growth condition of the type

$$|f(t, x)| \leqslant a(t) + b|x|^{p/q} \text{ for all } t \text{ and } x,$$

$$a \in L_q[0, 1] \text{ and } b > 0,\ 1/p + 1/q = 1.$$

(ii) $f(t, x) \geqslant 0$ for $x \geqslant 0$ and monotonically increasing with respect to x for almost all $t \in [0, 1]$.

(iii) There exists a constant α such that

$$|f(t, x) - f(t, y)| \leqslant \alpha|x - y| \text{ for all } x, y \geqslant 0 \text{ and } t \in [0, 1].$$

ASSUMPTIONS [D]

(i) For each $x \in C$, $K(x; s, t): [0, 1] X [0, 1] \to R$ is a non-negative function such that

$$\beta = \sup_{x \in C} \left[\int_0^1 \int_0^1 |K(x; s, t)|^p \right]^{1/p} < \infty.$$

(ii) If $x_1 \leqslant x_2$ in C then $K(x_1; s, t) \leqslant K(x_2; s, t)$ for all $s, t \in [0, 1]$.

(iii) $x \to K(x)$ is a compact map from C to $L(L_q, L_p)$ where $K(x)$ is the linear operator generated by $K(x; s, t)$.

THEOREM 3.2

Let the function $f(t, x)$ and kernel $K(x; s, t)$ satisfy the Assumptions [C] and [D] respectively. If $\alpha\beta < 1$, then the iterates $\{x_n\}$ of non-negative functions defined as

$$x_{n+1}(s) = x_0(s) + \int_0^1 K(x_n; s, t) f[t, x_n(t)]\, dt \tag{10}$$

converge to a non-negative solution of (9) in $L_p[0, 1]$.

Proof. We set $X = L_p[0, 1]$, $Y = L_q[0, 1]$ and C, D the cones of non-negative function in X and Y respectively. Define $K(x)$ and N as before. Equation (9) is then equivalent to the operator equation

$$x = x_0 + K(x) Nx. \tag{11}$$

Under assumptions [C] and [D], the operators $K(x)$ and N satisfy all the conditions of

Theorem 2.3 and hence the sequence defined by (10) converges to a solution $x(t)$ of (9).

We finally obtain a constructive proof of the existence of Chandrasekhar's H-equation

$$x(s) = 1 + x(s) \int_0^1 \left(\frac{s}{s+t}\right) \psi(t) x(t)\, dt. \tag{12}$$

This equation arises in the study of radiative transfer in a semi-infinite atmosphere and was first studied by Chandrasekhar in [4]. Subsequently a rigorous functional analytic proof was given by Stuart [14] and more recently by Joshi and Srikanth [6]. It is interesting to note that (12) becomes a very special case of generalized Hammerstein operator equation if we define operators $K(x)$ and N as

$$[K(x)](y)(s) = y(s) \int_0^1 \frac{s}{s+t} \psi(t) x(t) dt$$

and

$$[N(x)](s) = x(s)$$

in appropriate spaces.

The constructive result is a direct corollary of Theorem 2.3. This was originally obtained by Legget in [8] but we restate it here for the sake of completeness.

COROLLARY

Let ψ be a non-negative bounded measurable function on $[0, 1]$ satisfying $0 < \int_0^1 \psi(t) dt \leqslant \frac{1}{2}$. Define a sequence $\{x_n\}$ of non-negative functions as

$$x_n(s) = 1 + x_{n-1}(s) \int_0^1 \frac{s}{s+t} \psi(t) x_{n-1}(t) dt$$

$$x_0(s) = 1.$$

Then the monotonically increasing sequence $\{x_n(s)\}$ converges to a non-negative solution $x(s)$ of (12).

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Some theorems in temperature-rate-dependent theory of thermoelasticity

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Abstract. A theorem on the uniqueness of solutions, a generalised Hamilton's principle and a reciprocal theorem for dynamical mixed boundary value problems are obtained in the context of a linear anisotropic thermoelasticity theory which predicts a finite speed of propagation of thermal signals.

Keywords. Generalised thermoelasticity; second-sound effect; uniqueness of solution; Hamilton's principle; reciprocal theorem.

1. Introduction

The thermoelasticity theory first proposed by Green and Lindsay [9] and later by Suhubi [15] has aroused much interest in recent years. Unlike the coupled thermoelasticity theory [5], this theory includes the temperature rate among the constitutive variables and consequently predicts a finite speed for the propagation of thermal signals. Since thermal signals propagating with finite speeds have actually been observed in solids [1, 2], this theory is more general and physically more realistic than the coupled theory, and several problems revealing interesting phenomena which characterize this theory are contained in references [3, 4, 6–8].

The purpose of the present paper is to prove three main theorems, *viz.* (i) a uniqueness theorem (ii) a variational principle of Hamilton-type and (iii) a reciprocal theorem of Betti-Rayleigh-type for linearized anisotropic thermoelastic interactions, by employing the equations obtained in [9]. In §2 we summarize the governing equations and formulate an initial, mixed boundary value problem. In §3, we obtain the equation of energy balance in terms of the generalized free energy function introduced by Biot in [5], and in §4 we employ it, to establish a uniqueness theorem. Unlike in [9] and [10], our proof of the theorem rests on the positive definiteness of the energy function. In §5 we establish a Hamilton-type variational principle for the field equations. Whereas the corresponding principle in the coupled theory involves two functionals [14], our principle involves just one. In §6 we establish a reciprocal theorem which includes as special cases several important theorems obtained in earlier works, including the celebrated Betti-Rayleigh theorem of classical elasticity. We propose to report some significant applications of our reciprocal theorem in a separate communication.

2. Basic equations

In the context of the theory proposed by Green and Lindsay [9], the field equations for

linear thermoelastic interactions in a homogeneous and anisotropic solid are

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$t_{ij} = k_{ijrs} e_{rs} + a_{ij}(\theta + \alpha\dot{\theta}) \tag{2}$$

$$t_{ji,j} + \rho_0 F_i = \rho_0 \ddot{u}_i, \quad t_{ij} = t_{ji} \tag{3}$$

$$q_i = -\theta_0 (b_i \dot{\theta} + k_{ij}\theta_{,j}) \tag{4}$$

$$\rho_0 \eta = a + \mathrm{d}\theta + h\dot{\theta} - a_{ij} e_{ij} - b_i \theta_{,i} \tag{5}$$

$$\frac{\rho_0 r}{\theta_0} + k_{ij}\theta_{,ij} - \mathrm{d}\dot{\theta} - h\ddot{\theta} + a_{ij}\dot{e}_{ij} + 2b_i\dot{\theta}_{,i} = 0 \tag{6}$$

Unless stated to the contrary, the notation and the symbols in these equations and those to follow are as explained in [9].

With the aid of (1), equations (2) and (6) may be rewritten as

$$t_{ij} = k_{ijrs} u_{r,s} + a_{ij}(\theta + \alpha\dot{\theta}) \tag{7}$$

$$(\rho_0 r/\theta_0) + k_{ij}\theta_{,ij} - d\dot{\theta} - h\ddot{\theta} + a_{ij}\dot{u}_{i,j} + 2b_i\dot{\theta}_{,i} = 0 \tag{8}$$

Eliminating t_{ij} from (3) and (7), we obtain

$$k_{ijrs} u_{r,sj} + a_{ij}(\theta + \alpha\dot{\theta})_{,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \tag{9}$$

If we assume that at time $t = 0$, the body is at rest in its initial undeformed state and is at the reference temperature θ_0, zero temperature-velocity and zero entropy, then the following initial conditions are to be satisfied:

$$u_i(x, 0) = \dot{u}_i(x, 0) = \theta(x, 0) = \dot{\theta}(x, 0) = \eta(x, 0) = 0, \quad x \in B, \tag{10}$$

B being the initial configuration of the body.

Further, if we assume that for $t \geqslant 0$, (i) the surface forces are prescribed on a part ∂B_1 and the displacements on the remaining part ∂B_2 of the boundary surface ∂B of B, and (ii) the temperature is prescribed on a part ∂B_3 and the heat flux on the remaining part ∂B_4 of ∂B, then the following conditions are to be satisfied:

$$\left.\begin{aligned} t_{ji} n_j &= T_i(x, t), \ x \in \partial B_1 \\ u_i &= U_i(x, t), \ x \in \partial B_2 \end{aligned}\right\}, \quad t \geqslant 0 \tag{11}$$

$$\left.\begin{aligned} \theta &= \Theta(x, t), \ x \in \partial B_3 \\ q_i n_i &= Q(x, t), \ x \in \partial B_4 \end{aligned}\right\}, \quad t \geqslant 0 \tag{12}$$

In these equations $T_i(x, t)$, $U_i(x, t)$, $\Theta(x, t)$ and $Q(x, t)$ are prescribed functions in their respective domains. With $F_i(x, t)$ and $r(x, t)$ as prescribed functions for $x \in B$ and $t \geqslant 0$, the problem of determining the field variables $u_i(x, t)$, $\theta(x, t)$ and $t_{ij}(x, t)$ for $x \in B$ and $t > 0$ by solving the equations (7)–(9) under the conditions (10)–(12) constitutes an initial mixed boundary value problem. We will refer to this problem as the problem associated with the system: $\{(F_i, r); (T_i, U_i, \Theta, Q); (u_i, t_{ij}, \theta)\}$.

3. Energy equation

We now obtain the equation of energy balance in terms of the generalized free energy

function V introduced by Biot [5] through the equation

$$V = \varepsilon - \eta\theta_0. \tag{13}$$

The energy function considered by Green and Lindsay [9] is given by

$$\psi = \varepsilon - \eta\phi \tag{14}$$

and has the following explicit form:

$$\rho_0\psi = \sigma_0 - a(\theta + \alpha\dot{\theta}) - \tfrac{1}{2}\mathrm{d}\theta^2 - e\theta\dot{\theta} - \tfrac{1}{2}f\dot{\theta}^2 + \alpha b_i\dot{\theta}\theta_{,i} + a_{ij}e_{ij}(\theta + \alpha\dot{\theta}) + \tfrac{1}{2}\alpha k_{ij}\theta_{,i}\theta_{,j} + \tfrac{1}{2}k_{ijrs}e_{ij}e_{rs} \tag{15}$$

The scalar function ϕ appearing in (14) is given by ([9], eqn. (4.4)).

$$\phi = \theta_0 + \theta + \alpha\dot{\theta} + \beta\theta\dot{\theta} + \tfrac{1}{2}\gamma\dot{\theta}^2 \tag{16}$$

Eliminating ε from (13) and (14) and substituting for ψ, ϕ and η from (15), (16) and (5), we obtain the following quadratic form expansion for the Biot's energy function V:

$$\rho_0 V = \tfrac{1}{2}k_{ijrs}e_{ij}e_{rs} - b_i\theta\theta_{,i} + \tfrac{1}{2}\alpha k_{ij}\theta_{,i}\theta_{,j} + \tfrac{1}{2}\mathrm{d}\theta^2 + h\theta\dot{\theta} + \tfrac{1}{2}\alpha h\dot{\theta}^2 \tag{17}$$

In obtaining this expression we have made use of the initial conditions (10) and the restrictions on the material constants [[9], eqn (4.8)], and have neglected terms of third and higher degrees in the field variables.

If K denotes the kinetic energy per unit mass, i.e. if

$$K = \tfrac{1}{2}\dot{u}_i\dot{u}_i, \tag{18}$$

we obtain with the aid of (7) and (9) and the divergence theorem,

$$(\mathrm{d}/\mathrm{d}t)\int_m K\,\mathrm{d}m = \int_m [F_i\dot{u}_i - (1/\rho_0)\{k_{ijrs}u_{r,s} + a_{ij}(\theta + \alpha\dot{\theta})\}\dot{u}_{i,j}]\,\mathrm{d}m + \int_{\partial B} t_{ij}n_j\dot{u}_i\,\mathrm{d}A \tag{19}$$

where m is the mass included in B.

Eliminating $\int_m a_{ij}(\theta + \alpha\dot{\theta})\dot{u}_{i,j}\,\mathrm{d}m$ from (8) and (19) and using (4) and (17), we obtain

$$(\mathrm{d}/\mathrm{d}t)\int_m (K + V)\,\mathrm{d}m + N = \int_m [F_i\dot{u}_i + (r/\theta_0)(\theta + \alpha\dot{\theta})]\,\mathrm{d}m + \int_{\partial B}[t_{ij}\dot{u}_j - (q_i/\theta_0)(\theta + \alpha\dot{\theta})]\eta_i\,\mathrm{d}A \tag{20}$$

where

$$N = \int_B [(\mathrm{d}\alpha - h)\dot{\theta}^2 + 2b_i\dot{\theta}\theta_{,i} + k_{ij}\theta_{,i}\theta_{,j}]\,\mathrm{d}B \tag{21}$$

Equation (20) is the desired equation of energy balance. Equation (4.17) of Green and Lindsay [9] is an alternative form of a particular case of this equation.

It must be noted that because of the thermodynamical restrictions on the material constants obtained in [[9], eqn (4.9)], the integrand in the right side of (21) is non-negative and accordingly, we have

$$N \geqslant 0 \tag{22}$$

Further, we note that if the time derivatives are replaced by variations, (20) and (21) yield a variational principle analogous to the Biot's variational principle [[5], § 9] of coupled thermoelasticity.

4. Uniqueness theorem

We now employ (20) to establish the following

THEOREM

If the Biot's energy function V is positive definite, then there exists atmost one solution for the problem associated with the system:

$$\{(F_i, r); (T_i, U_i, \Theta, Q); (u_i, t_{ij}, \theta)\}.$$

Proof: To establish the theorem it is sufficient to show that for $F_i \equiv r \equiv 0$ in B and $T_i \equiv U_i \equiv \Theta \equiv Q \equiv 0$ in their respective domains on ∂B, the solution is trivial. For $F_i \equiv r \equiv 0$ and homogeneous boundary conditions, (20) simplifies to

$$(\mathrm{d}/\mathrm{d}t)\int_m (K+V)\,\mathrm{d}m = -N \tag{23}$$

Since the right side is non-positive, because of (22), it follows that $\int_m (K+V)\mathrm{d}m$ is a non-increasing function of time. At $t = 0$ we have $K = 0 = V$, in view of the initial conditions (10); consequently we should have

$$\int_m (K+V)\mathrm{d}m \leqslant 0 \quad \text{for all} \quad t \geqslant 0.$$

Since $K \geqslant 0$ by definition, it follows that

$$K + V = 0 \quad \text{or} \quad K = 0 = V \quad \text{for all} \quad t \geqslant 0, \tag{24}$$

provided V is positive definite.
Equation (24) readily yield the trivial solution

$$u_i(x, t) = \theta(x, t) = 0, \quad \text{for all} \quad x \in B,\ t \geqslant 0.$$

This completes the proof.

It may be mentioned here that Green and Lindsay [9] and Green [10] proved the above theorem by imposing restrictions on the individual material constants α, h and k_{ij}. The positive definiteness condition imposed by us on V takes care of all these restrictions and is in conformity with the nature of V in the coupled theory [13].

In the particular case when $\alpha = h = b_i = 0$, our energy equation (20) and the uniqueness theorem reduce to those obtained by Ionescu-Cazimir [12] in the context of the coupled theory.

5. Variational principle

We obtain below a generalized Hamilton's principle in the form of the following

THEOREM

If t_1 and t_2 are two arbitrary instants of time, then the field equations (8) and (9) form a set of necessary and sufficient conditions for the variational equation

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \Big[\int_m \{K - \psi + \theta(\partial\psi/\partial\theta) + F_i u_i + (\alpha/\theta_0) r\theta\}\,\mathrm{d}m + \int_{\partial B_2} T_i u_i \,\mathrm{d}s - (\alpha/\theta_0)\int_{\partial B_4} Q\theta\,\mathrm{d}A \Big] = 0 \tag{25}$$

to hold for arbitrary variations δu_i and $\delta\theta$ in u_i and θ respectively, which in addition to

being compatible with the kinematic constraints satisfy the conditions

$$\delta u_i(x, t_1) = \delta u_i(x, t_2) = \delta\theta(x, t_1) = \delta\theta(x, t_2) = 0, \quad x \in B \tag{26}$$

$$\delta u_i = 0 \text{ on } \partial B_1, \ \theta = 0 \text{ on } \partial B_3, \text{ for all } t \geqslant 0, \tag{27}$$

the functions F_i, r, $(\partial\psi/\partial\theta)$, T_i, Q and t being kept unchanged.

Proof: From the defn. of K, *viz.*, (18), we obtain

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m K \,\mathrm{d}m &= \int_m [\dot{u}_i \delta u_i \,|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{u}_i \delta u_i \,\mathrm{d}t] \,\mathrm{d}m \\ &= -\int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 \ddot{u}_i \delta u_i \,\mathrm{d}B,\end{aligned} \tag{28}$$

on using (26).
Also,

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (F_i u_i + (\alpha/\theta_0) r\theta) \,\mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 (F_i \delta u_i + (\alpha/\theta_0) r \delta\theta) \mathrm{d}B \tag{29}$$

and

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t [\int_{\partial B_2} T_i u_i \,\mathrm{d}A - \alpha \int_{\partial B_4} Q\theta \,\mathrm{d}A] = \int_{t_1}^{t_2} \mathrm{d}t [\int_{\partial B_2} T_i \delta u_i \,\mathrm{d}A \\ - \alpha \int_{\partial B_4} Q \delta\theta \,\mathrm{d}A]\end{aligned} \tag{30}$$

Further,

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (\psi - [\partial\psi/\partial\theta]\theta) \,\mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 ([\partial\psi/\partial e_{ij}] \delta e_{ij} + [\partial\psi/\partial\dot{\theta}] \delta\dot{\theta} \\ + [\partial\psi/\partial\theta_{,i}] \delta\theta_{,i}) \,\mathrm{d}B\end{aligned}$$

Substituting for ψ from (15) in the rightside of the above equation and simplifying the resulting expression with (1), (2), (4), (26), (27), the boundary conditions (11), (12) and the divergence theorem, we obtain

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (\psi - [\partial\psi/\partial\theta]\theta) \mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t [\int_{\partial B_2} T_i \delta u_i \mathrm{d}A - \int_B \{k_{ijrs} u_{r,sj} \\ + a_{ij}(\theta + \alpha\dot{\theta})_{,j}\} \delta u_i \mathrm{d}B \\ + \alpha \int_B (\mathrm{d}\dot{\theta} + h\ddot{\theta} - 2b_i \dot{\theta}_{,i} - a_{ij} \dot{u}_{i,j} \\ - k_{ij}\theta_{,ij}) \delta\theta \,\mathrm{d}B \\ - (\alpha/\theta_0) \int_{\partial B_4} Q \delta\theta \,\mathrm{d}A]\end{aligned} \tag{31}$$

Equations (28)–(31) yield

$$\begin{aligned}&\delta \int_{t_1}^{t_2} \mathrm{d}t [\int_m \{K - \psi + \theta(\partial\psi/\partial\theta) + F_i u_i + (\alpha/\theta_0) r\theta\} \mathrm{d}m \\ &\quad + \int_{\partial B_2} T_i u_i \,\mathrm{d}A - (\alpha/\theta_0) \int_{\partial B_4} Q\theta \,\mathrm{d}A] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_B [\{k_{ijrs} u_{r,sj} + a_{ij}(\theta + \alpha\dot{\theta})_{,j} + \rho_0 (F_i - \ddot{u}_i)\} \delta u_i \\ &\quad + \alpha\{(\rho_0/\theta_0) r + k_{ij}\theta_{,ij} - \mathrm{d}\dot{\theta} - h\ddot{\theta} + 2b_i \dot{\theta}_{,i} + a_{ij} \dot{u}_{i,j}\} \delta\theta] \,\mathrm{d}B\end{aligned}$$

From this equation, it is obvious that the variational equation (25) holds if and only if (8) and (9) hold. This completes the proof.

It is worth noting that while the generalized Hamilton's principle in coupled thermoelasticity involves *two* variational equations [14], the principle obtained above involves just *one* variational equation (*viz.*, equation (25)).

6. Reciprocal theorem

We now consider two initial, mixed boundary-value problems associated with the two systems

$$\{(F_i^{(\lambda)}, r^{(\lambda)}); (T_i^{(\lambda)}, U_i^{(\lambda)}, \Theta^{(\lambda)}, Q^{(\lambda)}); (u_i^{(\lambda)}, \theta^{(\lambda)})\}, \ \lambda = 1, 2$$

and suppose that $t_{ij}^{(\lambda)}$ and $q_i^{(\lambda)}$ are the corresponding stresses and heat flux respectively. For $\lambda, \mu = 1, 2, \lambda \neq \mu$, if

$$\begin{aligned} L_{\lambda\mu} = \int_B [\rho_0 (F_i^{(\lambda)} * \hat{u}_i^{(\mu)}) - (\rho_0/\theta_0)\{r^{(\lambda)} * \theta^{(\mu)} + \alpha(r^{(\lambda)} * \hat{\theta}^{(\mu)})\} \\ - 2b_i\{\theta_{,i}^{(\lambda)} * \hat{\theta}^{(\mu)} + \alpha(\theta_{,i}^{(\lambda)} * \hat{\hat{\theta}}^{(\mu)})\}]\,dB + \int_{\partial B_1} (T_i^{(\lambda)} * \hat{u}_i^{(\mu)})\,dA \\ - \int_{\partial B_2} U_i^{(\lambda)} * \hat{t}_{ij}^{(\mu)} n_j\,dA - (1/\theta_0)\int_{\partial B_3} \{\Theta^{(\lambda)} * q_i^{(\mu)} + \alpha(\Theta^{(\lambda)} * \hat{q}_i^{(\mu)})\} n_i\,dA \\ + \int_{\partial B_4} \{Q^{(\lambda)} * \theta^{(\mu)} + \alpha(Q^{(\lambda)} * \hat{\theta}^{(\mu)})\}\,dA, \end{aligned} \tag{32}$$

where

$$\begin{aligned} f * g &= \int_0^t f(x, t-\tau) g(x, \tau)\,d\tau \\ f * \hat{g} &= \int_0^t f(x, t-\tau)(\partial g/\partial\tau)(x, \tau)\,d\tau \\ f * \hat{\hat{g}} &= \int_0^t f(x, t-\tau)(\partial^2 g/\partial\tau^2)(x, \tau)\,d\tau, \end{aligned} \tag{33}$$

we prove the *reciprocal theorem* that

$$L_{12} = L_{21} \tag{34}$$

Proof: By hypothesis, the functions associated with the two problems considered are governed by (4) and (7)–(9) and the boundary conditions (11) and (12). Taking the Laplace transform with respect to t of these equations under the initial conditions (10), we obtain the following equations for the transformed functions

$$\text{In } B: \bar{q}_i^{(\lambda)} = -\theta_0[pb_i\bar{\theta}^{(\lambda)} + k_{ij}\bar{\theta}_{,j}^{(\lambda)}] \tag{35}$$

$$\bar{t}_{ij}^{(\lambda)} = k_{ijrs}\bar{u}_{r,s}^{(\lambda)} + a_{ij}(1 + \alpha p)\bar{\theta}^{(\lambda)} \tag{36}$$

$$\begin{aligned} &(\rho_0/\theta_0)\bar{r}^{(\lambda)} + k_{ij}\bar{\theta}_{,ij}^{(\lambda)} - dp\bar{\theta}^{(\lambda)} - hp^2\bar{\theta}^{(\lambda)} \\ &+ a_{ij}p\bar{u}_{i,j}^{(\lambda)} + 2b_i p\bar{\theta}_{,i}^{(\lambda)} = 0 \end{aligned} \tag{37}$$

$$k_{ijrs}\bar{u}_{r,sj}^{(\lambda)} + a_{ij}(1 + \alpha p)\bar{\theta}_{,j}^{(\lambda)} + \rho_0\bar{F}_i^{(\lambda)} = \rho_0 p^2\bar{u}_i^{(\lambda)} \tag{38}$$

and

$$\left.\begin{aligned} &\text{on } \partial B_1: \bar{t}_{ij}^{(\lambda)} n_j = \bar{T}_i^{(\lambda)} \\ &\text{on } \partial B_2: \bar{u}_i^{(\lambda)} = \bar{U}_i^{(\lambda)} \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &\text{on } \partial B_3: \bar{\theta}^{(\lambda)} = \bar{\Theta}^{(\lambda)} \\ &\text{on } \partial B_4: \bar{q}_i^{(\lambda)} n_i = \bar{Q}^{(\lambda)} \end{aligned}\right\} \tag{40}$$

In these equations $\bar{f} = \bar{f}(x, p)$ denotes the Laplace transform of $f = f(x, t)$.

From equations (36) and (38) we obtain

$$\begin{aligned} &\int_B [\rho_0(\bar{F}_i^{(1)}\bar{u}_i^{(2)} - \bar{F}_i^{(2)}\bar{u}_i^{(1)}) - (1 + \alpha p)a_{ij}(\bar{\theta}^{(1)}\bar{u}_{i,j}^{(2)} - \bar{\theta}^{(2)}\bar{u}_{i,j}^{(1)})]\,dB \\ &\quad = \int_B (\bar{t}_{ji}^{(2)}\bar{u}_i^{(1)} - \bar{t}_{ji}^{(1)}\bar{u}_i^{(2)})_{,j}\,dB \end{aligned} \tag{41}$$

From (35) and (37) we obtain

$$\int_B [p\{a_{ij}(\bar{u}_{i,j}^{(1)}\bar{\theta}^{(2)} - \bar{u}_{i,j}^{(2)}\bar{\theta}^{(1)}) + 2b_i(\bar{\theta}_{,i}^{(1)}\bar{\theta}^{(2)} - \bar{\theta}_{,i}^{(2)}\bar{\theta}^{(1)})\}$$
$$+ (\rho_0/\theta_0)(\bar{r}^{(1)}\bar{\theta}^{(2)} - \bar{r}^{(2)}\bar{\theta}^{(1)})]\,dB = (1/\theta_0)\int_B (\bar{q}_i^{(1)}\bar{\theta}^{(2)} - \bar{q}_i^{(2)}\bar{\theta}^{(1)})_{,i}\,dB \quad (42)$$

Converting the volume integrals in the right sides of (41) and (42) into surface integrals using the divergence theorem, applying the boundary conditions (39) and (40) and eliminating the common terms, we obtain the following Laplace transform form of the reciprocal theorem:

$$\int_B [\rho_0 p(\bar{F}_i^{(1)}\bar{u}_i^{(2)} - \bar{F}_i^{(2)}\bar{u}_i^{(1)}) - (\rho_0/\theta_0)(1+\alpha p)(\bar{r}^{(1)}\bar{\theta}^{(2)} - \bar{r}^{(2)}\bar{\theta}^{(1)})$$
$$- 2b_i p(1+\alpha p)(\bar{\theta}_{,i}^{(1)}\bar{\theta}^{(2)} - \bar{\theta}_{,i}^{(2)}\bar{\theta}^{(1)})]\,dB + p\int_{\partial B_1}(\bar{T}_i^{(1)}\bar{u}_i^{(2)} - \bar{T}_i^{(2)}\bar{u}_i^{(1)})\,dA$$
$$+ p\int_{\partial B_2}(\bar{t}_{ij}^{(1)}\bar{U}_i^{(2)} - \bar{t}_{ij}^{(2)}\bar{U}_i^{(1)})n_j\,dA + ([1+\alpha p]/\theta_0)$$
$$\times \int_{\partial B_3}(\bar{q}_i^{(1)}\bar{\Theta}^{(2)} - \bar{q}_i^{(2)}\bar{\Theta}^{(1)})n_i\,dA$$
$$+ (1+\alpha p)\int_{\partial B_4}(\bar{Q}^{(1)}\bar{\theta}^{(2)} - \bar{Q}^{(2)}\bar{\theta}^{(1)})\,dA = 0 \quad (43)$$

Inverting this equation with the aid of the convolution theorem for Laplace transforms, we obtain the desired result (34). This completes the proof.

It is interesting to note that the extra constant h, the presence of which is responsible for the inclusion of the temperature-acceleration term in the heat equation (8), does not influence the reciprocal theorem. If the material has a centre of symmetry at every point, but otherwise is anisotropic, we have $b_i = 0$, and our theorem then reduces to that obtained in [8]. If we set $\alpha = b_i = 0$ in (32), our reciprocal theorem (34) reduces to that in the classical coupled thermoelasticity theory. Ionescu-Cazimir [11] obtained a reciprocal theorem in coupled thermoelasticity for prescribed temperature on the entire boundary; obviously, his theorem corresponds to the case $\alpha = b_i = 0$ and $\partial B_3 = \partial B$ of our theorem (34).

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Dusty effects on pulsating viscous flow of two immiscible fluids between two parallel plates

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Abstract. The flow of two immiscible incompressible dusty viscous fluids between two parallel plates generated by a pulsating pressure gradient is investigated. Velocity fields for the fluid-particle system along with the expressions for the skin friction drag at the plates are obtained and studied graphically. It is found that there is an immediate response to pressure fluctuations in the first stream at low frequency range $0 < \sigma \leqslant 4$ being maximum at $\sigma = 4$. On the contrary, the second stream is more responsive to fluctuations at relatively higher frequencies. The maximum response in this case is shifted to $\sigma = 16$.

Keywords. Parallel plates; immiscible fluids; pulsatile flow; dusty effects; Stokes-Saffman layer.

1. Introduction

Problems on pulsating flows of fluids and problems associated with flows of immiscible fluids are of great theoretical and technical interest. Many authors like Phillips and Chiang [5], Bhattacharya and Nanda [1], Kant [3], Napolitano [4] and others have paid their attention to such flows. Recently, Gupta and Goyal [2] have studied the problem of pulsating flow of *n*-immiscible viscous fluids between two parallel plates. But they have not considered the effect of dust contamination on the flow although the presence of dust particles in the fluids can change the flow characteristics to a large extent.

In the present analysis, we have discussed dusty effects on the pulsatile flow of two immiscible viscous fluids between two parallel plates when the pressure gradient fluctuates about a steady mean. This study does not appear to have engaged the attention of researchers so far. Explicit expressions for the velocity fields of the fluid-particle system and skin friction at the plates have been derived and studied graphically for different density distributions and species of dust particles. It is observed that response to pressure gradient fluctuations is quicker in the first stream than in the second. Moreover, the magnitude of skin friction at the lower plate is much greater than that at the upper plate and its clean fluid value.

2. Governing equations

Consider unsteady motion of two viscous incompressible immiscible dusty fluids each occupying a height h between two stationary parallel flat plates kept at a distance $2h$

apart. Let μ_1, μ_2 and ρ_1, ρ_2 be respectively the coefficients of viscosity and densities of the fluids from lower to the upper plate. We choose a system of rectangular cartesian coordinates such that y' axis is perpendicular to the plates $y' = 0, y' = 2h$. The x' axis is chosen along the lower plate in the direction of pressure gradient. Let U_i' and V_i' be the velocities of the ith $(i = 1, 2)$ fluid and dust therein respectively along x' axis, N_i the number density of dust particles, m_i and r_i be respectively the mass and radius of a single dust particle, p' the isotropic pressure, v_i $(= \mu_i/\rho_i)$ the kinematic viscosity of the fluid, K_i $(= 6\pi\mu_i r_i$ for spherical particles) the Stokes coefficient of resistance, l_i $(= m_i N_i/\rho_i)$ the mass concentration and τ_i $(= m_i/K_i)$ the retardation time of dust particles in the ith fluid and t' the time. The plates are infinite in the x' and z' directions so that all the physical quantities are functions of y' and t' only. The dust particles are spherical and uniform in shape and size. The volume fraction of the solid particles is so small that the interaction between the solid particles is neglected. For the present problem, we have

$$U_i' = U_i'(y', t'),\ V_i' = V_i'(y', t'),\ N_i = N_{0i} = \text{constant}, \tag{1}$$

which satisfy the equation of continuity. Substituting (1) in equations of motion (Saffman [6]) and introducing the following dimensionless quantities

$$\left.\begin{aligned} &y = y'/h,\ U_i = hU_i'/v_i,\ V_i = hV_i'/v_i,\ t = v_i t'/h^2 \\ &Y_i = v_i\tau_i/h^2,\ k_i = \frac{h^2 K_i N_{0i}}{v_i\rho_i} = \frac{l_i}{Y_i},\ f(t) = \frac{h^3}{v_i^2}\left(-\frac{1}{\rho_i}\frac{\partial p_i}{\partial x'}\right), \end{aligned}\right\} \tag{2}$$

we obtain the following equations of motion

$$\frac{\partial U_i}{\partial t} = f(t) + \frac{\partial^2 U_i}{\partial y^2} + k_i(V_i - U_i) \tag{3}$$

$$Y_i\frac{\partial V_i}{\partial t} = U_i - V_i. \tag{4}$$

Eliminating V_i between (3) and (4) we derive

$$Y_i\frac{\partial^2 U_i}{\partial t^2} + (1 + l_i)\frac{\partial U_i}{\partial t} = [f(t) + Y_i f'(t)] + \left[\frac{\partial^2 U_i}{\partial y^2} + Y_i\frac{\partial^3 U_i}{\partial t\,\partial y^2}\right], \tag{5}$$

where $$f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}.$$

Equation (5) is to be solved under the following boundary conditions

$$\left.\begin{aligned} &U_1 = V_1 = 0 \text{ at } y = 0 \\ &U_2 = V_2 = 0 \text{ at } y = 2 \end{aligned}\right\}. \tag{6}$$

For uniform pulsating flow,

$$f(t) = x_{0i} + \mathrm{Re}\sum_{m=1}^{\infty} x_{mi}\exp(Jm\sigma t),\ J = \sqrt{-1}, \tag{7}$$

where Re denotes the real part. Coefficients x_{0i} and x_{mi} are Fourier constants of function $f(t)$. In view of (7) we take

$$U_i = U_0^{(i)} + \mathrm{Re}\sum_{m=1}^{\infty} U_m^{(i)}\exp(Jm\sigma t), \tag{8}$$

$$V_i = V_0^{(i)} + \mathrm{Re} \sum_{m=1}^{\infty} V_m^{(i)} \exp(Jm\sigma t), \tag{9}$$

where $U_0^{(i)}$, $V_0^{(i)}$, $U_m^{(i)}$, $V_m^{(i)}$ are functions of y only. Substituting (7) and (8) in (5) and comparing terms of the same family we get

$$(\mathrm{d}^2U_0^{(i)}/\mathrm{d}y^2) + x_{0i} = 0, \tag{10}$$

$$(\mathrm{d}^2U_m^{(i)}/\mathrm{d}y^2) - \lambda_i U_m^{(i)} + x_{mi} = 0, \tag{11}$$

where $$\lambda_i = \alpha_i + J\beta_i,\ \alpha_i = \frac{m^2 l_i Y_i \sigma^2}{1 + m^2\sigma^2 Y_i^2},\ \beta_i = \left[1 + \frac{l_i}{1 + m^2\sigma^2 Y_i^2}\right] m\sigma. \tag{12}$$

The conditions (6) are reduced to

$$\left.\begin{aligned} U_0^{(1)} = U_m^{(1)} = 0,\ V_0^{(1)} = V_m^{(1)} = 0 \text{ at } y = 0 \\ U_0^{(2)} = U_m^{(2)} = 0,\ V_0^{(2)} = V_m^{(2)} = 0 \text{ at } y = 2 \end{aligned}\right\}. \tag{13}$$

Neglecting surface tension at the interface and assuming that for moderate pulsation the shape of the interface will not change, the fluid velocities at the interface can be taken as

$$\frac{\nu_1}{h} U_i = A_1 + \mathrm{Re} \sum_{m=1}^{\infty} (Am)_1 \exp(Jm\sigma t) \text{ on } y = 1,\ i = 1, 2, \tag{14}$$

where A_1 and $(Am)_1$ are constants to be determined. Substituting (8) in (14) we derive

$$\frac{\nu_1}{h} U_0^{(1)} = \frac{\nu_2}{h} U_0^{(2)} = A_1 \text{ on } y = 1, \tag{15}$$

$$\frac{\nu_1}{h} U_m^{(1)} = \frac{\nu_2}{h} U_m^{(2)} = (Am)_1 \text{ on } y = 1. \tag{16}$$

3. Solution

3.1 *Steady part*

The solution of (10) is

$$U_0^{(i)} = -\tfrac{1}{2} x_{0i} y^2 + B_0^{(i)} y + C_0^{(i)},\ i = 1, 2, \tag{17}$$

which represents the parabolic flow for the steady part of the velocity distribution. Using (13), (15) and (17) we get

$$\left.\begin{aligned} B_0^{(1)} &= \frac{h}{\nu_1} A_1 + \tfrac{1}{2} x_{01},\ C_0^{(1)} = 0 \\ B_0^{(2)} &= -\frac{h}{\nu_2} A_1 + \tfrac{3}{2} x_{02},\ C_0^{(2)} = \frac{2h}{\nu_2} A_1 - x_{02} \end{aligned}\right\} \tag{18}$$

3.2 *Unsteady part*

Solving (11) we derive

$$U_m^{(i)} = B_i \cosh(\theta_i + J\phi_i) y + C_i \sinh(\theta_i + J\phi_i) y + \varepsilon_i,\ i = 1, 2, \tag{19}$$

where

$$\theta_i = \left[\frac{\alpha_i + (\alpha_i^2 + \beta_i^2)^{1/2}}{2}\right]^{1/2}, \quad \phi_i = \left[\frac{(\alpha_i^2 + \beta_i^2)^{1/2} - \alpha_i}{2}\right]^{1/2},$$

$$\varepsilon_i = \frac{x_{mi}(\alpha_i - J\beta_i)}{\alpha_i^2 + \beta_i^2}. \tag{20}$$

From (13), (16) and (19) B_i and C_i are evaluated as

$$B_1 = -\varepsilon_1, \; C_1 = \frac{(h/v_1)(Am)_1 + \varepsilon_1\{\cosh(\theta_1 + J\phi_1) - 1\}}{\sinh(\theta_1 + J\phi_1)},$$

$$B_2 = \frac{[\{(h/v_2)(Am)_1 - \varepsilon_2\}\sinh 2(\theta_2 + J\phi_2) + \varepsilon_2 \sinh(\theta_2 + J\phi_2)]}{\sinh(\theta_2 + J\phi_2)},$$

$$C_2 = \frac{[\{\varepsilon_2 - (h/v_2)(Am)_1\}\cosh 2(\theta_2 + J\phi_2) - \varepsilon_2 \cosh(\theta_2 + J\phi_2)]}{\sinh(\theta_2 + J\phi_2)}. \tag{21}$$

From (8), (17) and (19) the complete velocity distribution in the ith ($i = 1, 2$) fluid is obtained as

$$U_i = [U_i]_0 + \mathrm{Re} \sum_{m=1}^{\infty} \{B_i \cosh(\theta_i + J\phi_i)y + C_i \sinh(\theta_i + J\phi_i)y + \varepsilon_i\} \exp(Jm\sigma t). \tag{22}$$

Substituting (8) and (9) in (4), comparing terms and using (22) we get

$$V_i = [V_i]_0 + \mathrm{Re} \sum_{m=1}^{\infty} \left(\frac{1 - Jm\sigma Y_i}{1 + m^2\sigma^2 Y_i^2}\right)\{B_i \cosh(\theta_i + J\phi_i)y + C_i \sinh(\theta_i + J\phi_i)y + \varepsilon_i\} \exp(Jm\sigma t) \tag{23}$$

$$U_i - V_i = \mathrm{Re} \sum_{m=1}^{\infty} \left\{\frac{m\sigma Y_i(m\sigma Y_i + J)}{1 + m^2\sigma^2 Y_i^2}\right\}\{B_i \cosh(\theta_i + J\phi_i)y + C_i \sinh(\theta_i + J\phi_i)y + \varepsilon_i\} \exp(Jm\sigma t) \tag{24}$$

where

$$[U_i]_0 = [V_i]_0 = -\tfrac{1}{2}x_{0i}y^2 + B_0^{(i)}y + C_0^{(i)}. \tag{25}$$

From (25) it is seen that steady parts of the velocity functions are identical. From (22) and (23) it is evident that the amplitude of the unsteady part of the dust velocity is less, being reduced by a factor $1/(1 + m^2\sigma^2 Y_i^2)^{1/2}$ from that of fluid velocity.

4. Determination of interface velocities

We consider the continuity of shear at the interface. This gives

$$\left(g_{1,2}\frac{\partial U_1}{\partial y}\right)_{y=1} = \left(\frac{\partial U_2}{\partial y}\right)_{y=1} \tag{26}$$

where $g_{1,2} = (\mu_1/\mu_2)\cdot(v_1/v_2)$ is the non-dimensional parameter representing the interplay between the viscous and inertia forces of the fluids. Inserting (22) in (26), using

(18) and (21) and equating terms of same family we derive

$$h\left(\frac{1}{\nu_2}+\frac{g_{1,2}}{\nu_1}\right)A_1 = \tfrac{1}{2}(x_{02}+g_{1,2}x_{01}) \tag{27}$$

$$g_{1,2}(\theta_1+J\phi_1)\left\{\frac{h}{\nu_1}(Am)_1-\varepsilon_1\right\}\coth(\theta_1+J\phi_1)+(\theta_2+J\phi_2).$$
$$\times\left[\left\{\frac{h}{\nu_2}(Am)_1-\varepsilon_2\right\}\coth(\theta_2+J\phi_2)+\varepsilon_2\operatorname{cosech}(\theta_2+J\phi_2)\right]$$
$$= -g_{1,2}\varepsilon_1(\theta_1+J\phi_1)\operatorname{cosech}(\theta_1+J\phi_1). \tag{28}$$

These relations provide constants A_1 and $(Am)_1$ which determine the interface velocity.

5. Skin friction

The skin friction at the plates are

$$T_1 = \left[\frac{\partial U_1}{\partial y}\right]_{y=0} = [T_1]_0 + \mathrm{Re}\sum_{m=1}^{\infty}(\theta_1+J\phi_1)C_1\exp(Jm\sigma t) \tag{29}$$

$$T_2 = \left[\frac{\partial U_2}{\partial y}\right]_{y=2h} = [T_2]_0 + \mathrm{Re}\sum(\theta_2+J\phi_2)\{B_2\sinh(\theta_2+J\phi_2)$$
$$+C_2\cosh 2(\theta_2+J\phi_2)\}\exp(Jm\sigma t) \tag{30}$$

where $[T_1]_0 = B_0^{(1)}$, $[T_2]_0 = -2x_{02}+B_0^{(2)}$ (31)

6. Mean sectional velocities

The sectional mean velocities $U^*_{iMV(t)}$, $V^*_{iMV(t)}$ of the fluid particle system (for $i = 1, 2$) are

$$U^*_{1MV(t)} = \int_0^1 U_1(y,t)\,dy$$
$$= [U^*_{1MV}]_0 + \mathrm{Re}\sum_{m=1}^{\infty}\left[\frac{1}{(\theta_1+J\phi_1)}\{B_1\sinh(\theta_1+J\phi_1)\right.$$
$$\left.+C_1(\cosh(\theta_1+J\phi_1)-1)\}+\varepsilon_1\right]\exp(Jm\sigma t) \tag{32}$$

$$V^*_{1MV(t)} = \int_0^1 V_1(y,t)\,dy$$
$$= [V^*_{1MV}]_0 + \mathrm{Re}\sum_{m=1}^{\infty}\left\{\frac{1-Jm\sigma Y_1}{1+m^2\sigma^2Y_1^2}\right\}$$
$$\times\left[\frac{1}{(\theta_1+J\phi_1)}\{B_1\sinh(\theta_1+J\phi_1)\right.$$
$$\left.+C_1(\cosh(\theta_1+J\phi_1)-1)\}+\varepsilon_1\right]\exp(Jm\sigma t), \tag{33}$$

$$U^*_{2MV(t)} = \int_1^2 U_2(y,t)\,dy$$
$$= [U^*_{2MV}]_0 + \mathrm{Re}\sum_{m=1}^{\infty}\left[\frac{2}{(\theta_2+J\phi_2)}\{B_2\cosh\tfrac{3}{2}(\theta_2+J\phi_2)\right.$$
$$\left.+C_2\sinh\tfrac{3}{2}(\theta_2+J\phi_2)\}\sinh\frac{(\theta_2+J\phi_2)}{2}+\varepsilon_2\right]\exp(Jm\sigma t) \tag{34}$$

$$V^*_{2MV(t)} = \int_1^2 V_2(y,t)\,dy$$

$$= [V^*_{2MV}]_0 + \mathrm{Re}\sum_{m=1}^{\infty}\left\{\frac{1-Jm\sigma Y_2}{1+m^2\sigma^2 Y_2^2}\right\}$$

$$\times\left[\frac{2}{\theta_2+J\phi_2}\left\{B_2\cosh\tfrac{3}{2}(\theta_2+J\phi_2)\right.\right.$$

$$\left.\left.+C_2\sinh\tfrac{3}{2}(\theta_2+J\phi_2)\right\}\sinh\frac{(\theta_2+J\phi_2)}{2}+\varepsilon_2\right]\exp(Jm\sigma t) \qquad (35)$$

where

$$\left.\begin{aligned}[U^*_{1MV}]_0 &= [V^*_{1MV}]_0 = -\frac{x_{01}}{6}+\frac{B_0^{(1)}}{2}\\ [U^*_{2MV}]_0 &= [V^*_{2MV}]_0 = -\tfrac{7}{6}x_{02}+\tfrac{3}{2}B_0^{(2)}+C_0^{(2)}\end{aligned}\right\}. \qquad (36)$$

7. Boundary layers

When θ_i and ϕ_i are large

$$\left.\begin{aligned}&\cosh(\theta_i+J\phi_i)\\ &\sinh(\theta_i+J\phi_i)\end{aligned}\right\}\simeq\tfrac{1}{2}\exp(\theta_i+J\phi_i),\ i=1,2. \qquad (37)$$

Introducing the boundary layer coordinate

$\eta = 1-y$, on the interface,

$\eta = 2-y$, on the upper plate,

in (22)–(23) and taking real parts we get

$$U_i(y,t) = [U_i(y)]_0 + \sum_{m=1}^{\infty}\left[\frac{\exp(-\theta_i\eta)}{(\Sigma_i\theta_i^2+\Sigma_i\phi_i^2)^{1/2}}\{P_i\cos(\phi_i\eta-\psi-m\sigma t)\right.$$

$$\left.-Q_i\sin(\phi_i\eta-\psi-m\sigma t)\}+\{a_i\cos m\sigma t-b_i\sin m\sigma t\}\right] \qquad (38)$$

$$V_i(y,t) = [V_i(y)]_0 + \sum_{m=1}^{\infty}\frac{1}{1+m^2\sigma^2Y_i^2}\left[\frac{(1+m^2\sigma^2Y_i^2)^{1/2}\exp(-\theta_i\eta)}{(\Sigma_i\theta_i^2+\Sigma_i\phi_i^2)^{1/2}}\right.$$

$$\times\{P_i\cos(\phi_i\eta-\psi_i-m\sigma t)-Q_i\sin(\phi_i\eta-\psi_i-m\sigma t)\}$$

$$\left.+\{L_i\cos m\sigma t+M_i\sin m\sigma t\}\right]. \qquad (39)$$

Equations (38)–(39) show the existence of multiple boundary layers of thickness

$$\delta_i = \left\{\frac{\alpha_i+(\alpha_i^2+\beta_i^2)^{1/2}}{2}\right\}^{-1/2},\ i=1,2, \qquad (40)$$

in the first and second dusty fluids respectively. These may be identified as the Stokes-Saffman layers and can be viewed either as a Classical Stokes layer modified by the presence of dust or as Saffman layer modified by pulsation. Outside the boundary

layers, in cores given by $\eta > \delta_i$, the exponential terms in (38)–(39) vanish and the velocity fields reduce to

$$U_i(y, t) = [U_i(y)]_0 + \sum_{m=1}^{\infty} [a_i \cos m\sigma t - b_i \sin m\sigma t], \; i = 1, 2$$

$$V_i(y, t) = [V_i(y)]_0 + \sum_{m=1}^{\infty} [L_i \cos m\sigma t + M_i \sin m\sigma t]. \tag{41}$$

7.1 *Saffman layer*

When mass concentration l_i of dust particles is very large and the size of dust particles are greater than a critical value ($Y_i > 1/m\sigma$), then $\alpha_i \gg \beta_i$. Hence, $\theta_i \simeq (\alpha_i)^{1/2}$, $\phi_i \simeq 0$ for $i = 1, 2$. Then from (22)–(23) we get

$$U_i(y, t) = [U_i(y)]_0 + \sum_{m=1}^{\infty} \left[\frac{\exp[-(\alpha_i)^{1/2}\eta]}{(\Sigma_i \alpha_i)} \{R_i \cos m\sigma t - S_i \sin m\sigma t\} + \right.$$

$$\left. + (a_i \cos m\sigma t - b_i \sin m\sigma t) \right], \tag{42}$$

$$V_i(y, t) = [V_i(y)]_0 + \sum_{m=1}^{\infty} \frac{1}{(1 + m^2\sigma^2 Y_i^2)^{1/2}} \left[\frac{(1 + m^2\sigma^2 Y_i^2)^{1/2}}{(\Sigma_i \alpha_i)} \times \right.$$

$$\times \exp[-(\alpha_i)^{1/2}\eta] \{R_i \cos(m\sigma t + x_i) + S_i \sin(m\sigma t + x_i)\} +$$

$$\left. + \{L_i \cos m\sigma t + M_i \sin m\sigma t\} \right]. \tag{43}$$

Equation (42)–(43) demonstrate the existence of thin boundary layers of thickness

$$p_i \simeq \frac{1}{(k_i)^{1/2}} \left(1 + \frac{1}{2m^2\sigma^2 Y_i^2}\right), \; i = 1, 2 \tag{44}$$

near the interface and the upper plate respectively. These may be identified as the modified (by the term $1/2m^2\sigma^2 Y_i^2$) Saffman layers. Thickness of these layers decreases with increase of m and σ. Boundary layers arising due to higher modes of pulsation ($m > 1$) are confined inside the boundary layer due to the fundamental mode ($m = 1$). Increasing values of number density or decreasing values of size of particles tend to decrease the boundary layer thickness.

7.2 *Stokes layers*

When frequency σ of pulsation is very large and mass concentration l_i of the particles is small, then $\beta_i \gg \alpha_i$. Hence from (20),

$$\theta_i \simeq c_i, \; \phi_i \simeq d_i$$

where

$$\left.\begin{aligned} c_i &= (m\sigma/2)^{1/2} \left[1 + \frac{k_i}{2m\sigma}\left(1 + \frac{1}{m\sigma Y_i}\right)\right], \; i = 1, 2 \\ d_i &= (m\sigma/2)^{1/2} \left[1 + \frac{k_i}{2m\sigma}\left(1 - \frac{1}{m\sigma Y_i}\right)\right] \end{aligned}\right\} \tag{45}$$

Hence from (22)–(23) we derive

$$U_i(y, t) = [U_i(y)]_0 + \sum_{m=1}^{\infty} \left[\frac{\exp(-c_i\eta)}{(\Sigma_i c_i^2 + \Sigma_i d_i^2)} \{F_i \cos(d_i\eta - \zeta - m\sigma t) - G_i \sin(d_i\eta - \zeta - m\sigma t)\} + \{a_i \cos m\sigma t - b_i \sin m\sigma t\} \right] \tag{46}$$

$$V_i(y, t) = [V_i(y)]_0 + \sum_{m=1}^{\infty} \frac{1}{1 + m^2\sigma^2 Y_i^2} \left[\frac{(1 + m^2\sigma^2 Y_i^2)^{1/2} \exp(-c_i\eta)}{\{\Sigma_i c_i^2 + \Sigma_i d_i^2\}^{1/2}} \times \{F_i \cos(d_i\eta - \xi_i - m\sigma t) - G_i \sin(d_i\eta - \xi_i - m\sigma t)\} + \{L_i \cos m\sigma t + M_i \sin m\sigma t\} \right] \tag{47}$$

Expressions (46)–(47) demonstrate the existence of boundary layers of thickness

$$q_i \simeq (2/m\sigma)^{1/2} \left[1 - \frac{k_i}{2m\sigma} \right], \; i = 1, 2 \tag{48}$$

near the interface and the upper plate respectively. These may be identified as the modified (by the term $k_i/2m\sigma$) Stokes layers. From (48) it is seen that the presence of dust particles decreases the thickness of the Stokes layers.

8. Flow under harmonic pressure gradient

We consider the particular case when the pressure gradient is given by

$$f(t) = x_{0i} + x_{ci} \cos \sigma t, \; i = 1, 2. \tag{49}$$

In this case, the sectional mean velocities and skin friction are obtained as

$$\left.\begin{aligned} U^*_{iMV(t)} &= [U^*_{iMV}]_0 + R_{iMV} \cos(\sigma t - \theta_{iMV}) \\ V^*_{iMV(t)} &= [V^*_{iMV}]_0 + R_{iD} \cos(\sigma t - \theta_{iD}), \; i = 1, 2, \\ T^*_i(t) &= [T^*_i]_0 + R_{iSF} \cos(\sigma t - \theta_{iSF}) \end{aligned}\right\} \tag{50}$$

where (R_{iMV}, θ_{iMV}), (R_{iD}, θ_{iD}) respectively denote the amplitude coefficients and phase lag of the sectional mean velocities from the wave of pressure gradient in the dusty fluids. Similarly, (R_{iSF}, θ_{iSF}) denote the corresponding quantities for the shear stresses at the plates in the direction of flow of the fluids.

The explicit forms of various coefficients in §§(7) and (8) are not given due to short of space.

9. Discussion

Figure 1 reveals that increasing values of the density distribution of particles amplify the response of the dusty mode in the first stream of low frequency range $0 < \sigma \leqslant 4$. The maximum response occurs at $\sigma = 4$ which is higher and sharper than that of the clean mode. This amplification is due to energy transfer from dust components to fluid particles at low frequencies and near the critical frequency. While light particles move

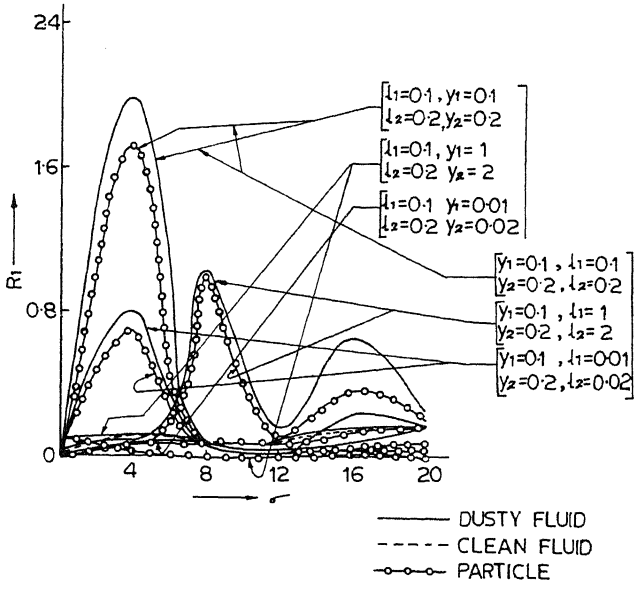

Figure 1. Variation of amplitude of sectional mean velocity of first dusty fluid with frequency.

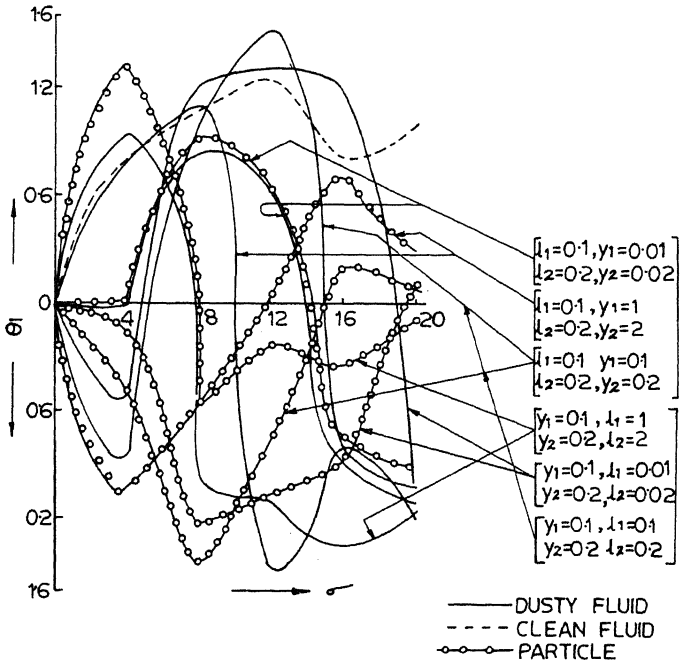

Figure 2. Variation of phase lag of sectional mean velocity in the first dusty fluid with frequency.

with the fluid, heavy particles show tendency towards sedimentation. Moreover, large density distribution or size of particles produces phase lag in the dusty mode in the first stream towards low frequencies while particles of low and moderate size or density distribution exert opposite influence (figure 2).

In the second stream (figure 3) there is almost no response to fluctuations at low frequencies. The maximum response occurs at $\sigma = 16$. As the density distribution of particles decreases, the response frequency increases and vice versa. Increasing values of the density of particle distribution enhance the phase lag of the dusty mode in this stream at low frequencies. Similar values of particle size display opposite behaviour (figure 4).

The amplitude of skin friction at the lower plate is much higher than at the upper plate and its clean fluid value (figure 5 and 6). While clean drag in both the streams always lags behind the pulsation, the dusty drag does so only partially (figures 7 and 8).

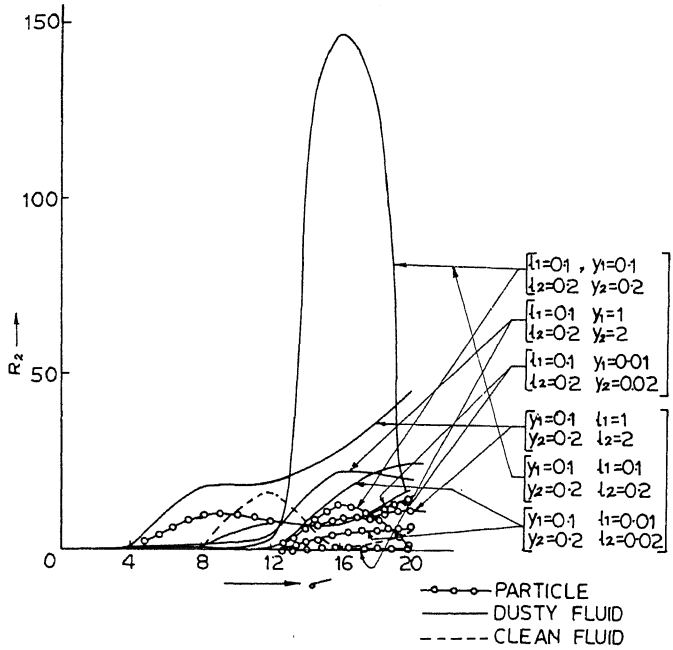

Figure 3. Variation of amplitude of sectional mean velocity of second dusty fluid with frequency.

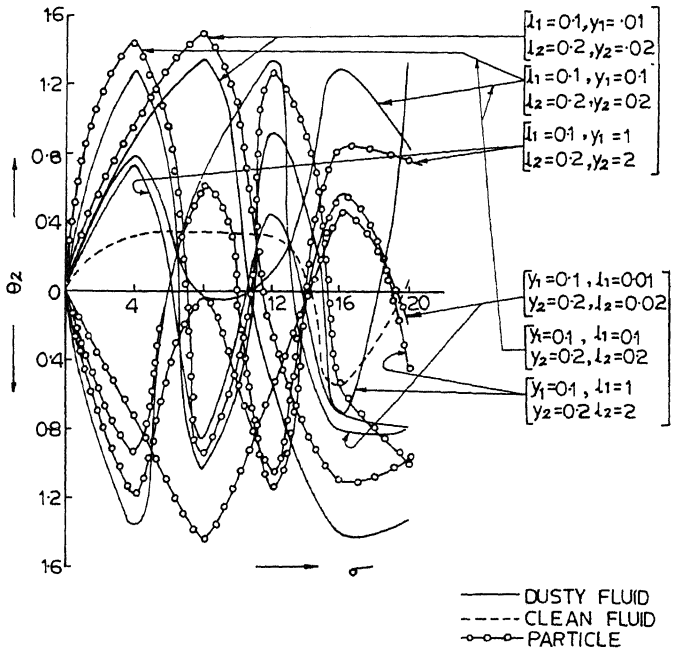

Figure 4. Variation of phase lag of sectional mean velocity in the second dusty fluid with frequency.

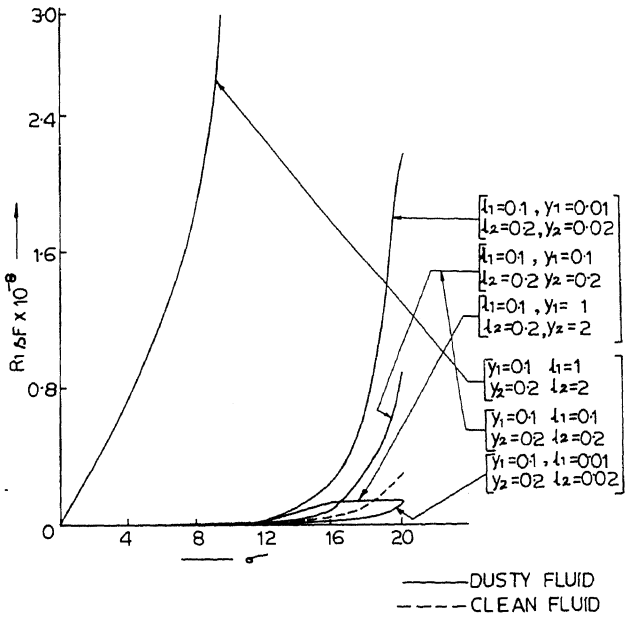

Figure 5. Variation of amplitude of skin friction at the lower plate with frequency.

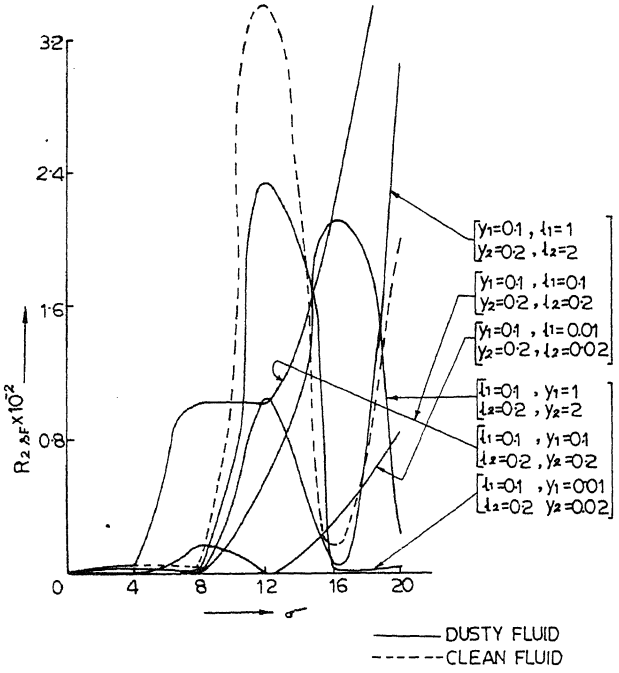

Figure 6. Variation of amplitude of skin friction at the upper plate with frequency.

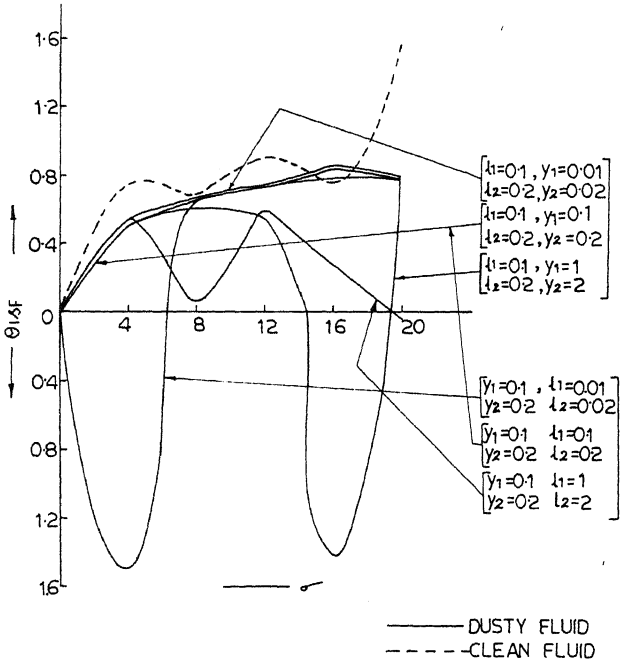

Figure 7. Variation of phase angle of skin friction at the lower plate with frequency.

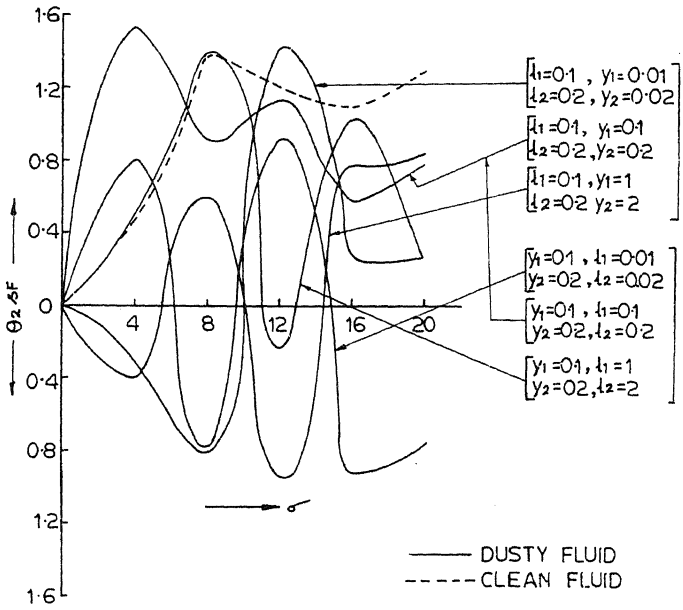

Figure 8. Variation of phase angle of skin friction at the upper plate with frequency.

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MHD slow motion past a sphere

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Abstract. The slow motion of an incompressible, viscous electrically conducting fluid, in the presence of a uniform aligned magnetic field, past a sphere is studied. Solutions obtained by Chester, using Stokes' approximations, and by Blerkom and Ludford, using Oseen' approximations, are reviewed. Expressions for stream functions are obtained for MHD Stokes' flow and Oseen' flow respectively.

Keywords. Slow motion; magnetohydrodynamics.

1. Introduction

Parallel to the Stokes' and Oseen' approximate studies of slow motion in ordinary fluid dynamics (OFD), efforts have been made to study the effects of magnetic field on the flow of a viscous electrically conducting fluid past bodies at low values of viscous and magnetic Reynolds numbers. Magnetohydrodynamics (MHD) flow past a sphere in the presence of an externally applied magnetic field has been studied by Chester [5], Blerkom [2], Ludford [7] and Mathon and Ranger [9]. The case in which the magnetic field originates in the body itself has been investigated by Ludford and Murray [8], Chopra and Singer [6], Barthel and Lykoudis [1] and others. MHD flow past a circular cylinder has been studied by Yosinobu and Kakutani [12], Bramley [3], Shanti Swarup and Sinha [11] and others. Some of these works have been summarized in the book by Cabannes [4].

This shows that the work of Chester [5] for MHD Stokes' flow and that of Blerkom [2] and Ludford [7] for MHD Oseen' flow is of fundamental interest and need to be reviewed. It is noted that the expressions for the dimensionless functions ϕ_1, ϕ_2, velocity v and the pressure p are misprinted in the paper of Chester [5], whereas slight correction in the expression for integrand, in calculating drag on the sphere, in Ludford [7] paper is needed. Further, we have added the expressions for the stream function in both the cases.

2. Formulation of the problem

Consider the streaming motion of an incompressible, viscous, electrically conducting fluid past a sphere, which is assumed to have the same magnetic permeability as the fluid. At infinity, the streaming motion and the magnetic field are assumed to be uniform and their directions are parallel to the positive x-axis. The flow is assumed to be steady.

The equations governing the motion in non-dimensional form, in the usual way, are

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\mathrm{Re}(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathrm{Re}(\hat{i}\cdot\nabla)\mathbf{v} = -\nabla p + \nabla^2\mathbf{v} + \mathrm{Re}_h^2\,\mathbf{J}\times\mathbf{H} \tag{2}$$

$$\mathrm{Re}_\sigma \nabla\times(\mathbf{v}\times\mathbf{H}) + \mathrm{Re}_\sigma\nabla\times(\hat{i}\times\mathbf{H}) = \nabla\times(\nabla\times\mathbf{H}) \tag{3}$$

where $$\mathbf{J} = \mathbf{v}\times\mathbf{H} + \hat{i}\times\mathbf{H} \tag{4}$$

with the boundary conditions

$$\left.\begin{aligned} r = 1 &: \mathbf{v} = -\hat{i} \\ r = \infty &: \mathbf{v} = 0,\ p = 0 \end{aligned}\right\} \tag{5}$$

where

$$\left.\begin{aligned} &\mathrm{Re} = \rho U_\infty a/\mu \text{ (viscous Reynolds number)} \\ &\mathrm{Re}_h = \mu_e H_\infty a\sqrt{\sigma/\mu} \text{ (Hartmann number)} \\ &\mathrm{Re}_\sigma = \mu_e \sigma a U_\infty \text{ (Magnetic Reynolds number)} \end{aligned}\right\} \tag{6}$$

ρ is the mass density, σ is the electrical conductivity, μ_e is the magnetic permeability, and μ is the viscosity of the fluid. U_∞ is the speed of the uniform stream at infinity parallel to the x-axis, H_∞ is the intensity of the applied aligned external magnetic field and a is the radius of the sphere.

3. MHD Stokes' flow

As in OFD Stokes' flow, we assume that Re is small. In addition we assume that Re_σ is also small. In many practical problems the first condition implies the second but the converse need not be true (MHD boundary layers).

Neglecting the terms containing Re_σ in (3), we get

$$\nabla\times(\nabla\times\mathbf{H}) = 0 \tag{7}$$

This shows that in MHD Stokes' flow the magnetic field is independent of the fluid velocity. The problem is now reduced to the solution of (1) and (2) for $\mathbf{v}$ and p, with $\mathbf{J}$ given by (4) and with a prescribed value of $\mathbf{H} = \hat{i}$.

Neglecting the terms containing Re in (2), the MHD Stokes' equations are

$$\nabla\cdot\mathbf{v} = 0 \tag{8}$$

$$-\nabla p + \nabla^2\mathbf{v} - \mathrm{Re}_h^2\{\mathbf{v} - (\mathbf{v}\cdot\hat{i})\hat{i}\} = 0 \tag{9}$$

Chester [5] assumed the following forms for $\mathbf{v}$ and p.

$$\mathbf{v} = \exp(\mathrm{Re}_h x)\nabla\phi_1 + \exp(-\mathrm{Re}_h x)\nabla\phi_2, \tag{10}$$

$$p = \mathrm{Re}_h\left(\exp(\mathrm{Re}_h x)\frac{\partial\phi_1}{\partial x} - \exp(-\mathrm{Re}_h x)\frac{\partial\phi_2}{\partial x}\right), \tag{11}$$

where ϕ_1 and ϕ_2 are dimensionless functions to be determined. As $\mathrm{Re}_h \to 0$, both ϕ_1 and ϕ_2 are of the order of $1/\mathrm{Re}_h$ so that the limiting value of the pressure is non-zero, which

should be the case for non-magnetic Stokes' flow. Moreover, the singular terms in (10) cancel each other, so that a finite limiting value of the velocity is also obtained.

With these forms of $\mathbf{v}$ and p, (8) and (9) are satisfied provided that

$$\nabla^2\phi_1 + \mathrm{Re}_h\frac{\partial\phi_1}{\partial x} = 0, \tag{12}$$

$$\nabla^2\phi_2 - \mathrm{Re}_h\frac{\partial\phi_2}{\partial x} = 0. \tag{13}$$

The general solutions of ϕ_1 and ϕ_2, keeping the symmetry in view, which vanish at infinity are obtained as, Milne-Thomson [10],

$$\phi_1 = \exp((-\mathrm{Re}_h/2)x)\sum_{n=0}^{\infty} C_n\left(\frac{\partial}{\partial x}\right)^n\left(\frac{1}{r}\exp((-\mathrm{Re}_h/2)r)\right), \tag{14}$$

$$\phi_2 = \exp((\mathrm{Re}_h/2)x)\sum_{n=0}^{\infty} (-1)^{n+1}C_n\left(\frac{\partial}{\partial x}\right)^n\left(\frac{1}{r}\exp((-\mathrm{Re}_h/2)r)\right). \tag{15}$$

These expressions and the corresponding expressions for $\mathbf{v}$ and p from (10) and (11) are misprinted in Chester [5].

Let

$$\mathbf{v} = \{v_1, v_2, v_3\},\ \mathbf{r} = \{x_1, x_2, x_3\} \tag{16}$$

Then from (10), taking $x = x_1$, substituting ϕ_1 and ϕ_2 from (14) and (15) and retaining small quantities of Re_h to the first order, we get

$$v_1 = 2C_1\left(\frac{3x_1^2}{r^5} - \frac{1}{r^3}\right) + \mathrm{Re}_h\left[C_2\left(\frac{1}{r^3} + \frac{6x_1^2}{r^5} - \frac{15x_1^4}{r^7}\right) - C_0\left(\frac{1}{r} + \frac{x_1^2}{r^3} - \frac{\mathrm{Re}_h}{2}\right)\right] + 0(\mathrm{Re}_h^2), \tag{17}$$

$$v_2 = \frac{6C_1x_1x_2}{r^5} + \mathrm{Re}_h\left[-C_0\frac{x_1x_2}{r^3} + C_2\left(\frac{3x_1x_2}{r^5} - \frac{15x_1^3x_2}{r^7}\right)\right] + 0(\mathrm{Re}_h^2), \tag{18}$$

$$v_3 = \frac{6C_1x_1x_3}{r^5} + \mathrm{Re}_h\left[-C_0\frac{x_1x_3}{r^3} + C_2\left(\frac{3x_1x_3}{r^5} - \frac{15x_1^3x_3}{r^7}\right)\right] + 0(\mathrm{Re}_h^2). \tag{19}$$

Applying the boundary conditions (5) and equating to zero the like powers of x_1, to a first order in Re_h, we find

$$\left.\begin{aligned} C_0\,\mathrm{Re}_h &= \frac{6}{8-3\mathrm{Re}_h} \approx \frac{3}{4}\left(1+\frac{3}{8}\mathrm{Re}_h\right),\\ C_1 &= \frac{1}{8-3\mathrm{Re}_h} \approx \frac{1}{8}\left(1+\frac{3}{8}\mathrm{Re}_h\right),\\ C_2 &= 0. \end{aligned}\right\} \tag{20}$$

Hence,

$$v_i = \frac{3}{4}\left(1+\frac{3}{8}\mathrm{Re}_h\right)\frac{x_1 x_i}{r^3}\left(\frac{1}{r^2}-1\right) - \frac{1}{4}\left(1+\frac{3}{8}\mathrm{Re}_h\right)\left(\frac{3}{r}+\frac{1}{r^3}-\frac{3\mathrm{Re}_h}{2}\right)\delta_{i1} + 0(\mathrm{Re}_h^2) \tag{21}$$

and

$$p = -\frac{3}{2}\left(1+\frac{3}{8}\mathrm{Re}_h\right)\frac{x_1}{r^3} + 0(\mathrm{Re}_h^2), \tag{22}$$

where δ_{i1} is the Kronecker delta.

3.1 *Drag in* MHD *Stokes' flow*

The total drag experienced by the sphere is calculated as

$$D = D_S\left[1+\frac{3}{8}\mathrm{Re}_h + 0(\mathrm{Re}_h^2)\right] \tag{23}$$

where, $D_S = 6\pi\mu U_\infty a$ (Stokes' drag in OHD). (24)

In fact Chester calculated the drag upto the third power of Re_h and reported that

$$D = D_S\left\{1+\frac{3}{8}\mathrm{Re}_h + \frac{7}{960}\mathrm{Re}_h^2 - \frac{43}{7680}\mathrm{Re}_h^3 + 0(\mathrm{Re}_h^4)\right\} \tag{25}$$

However, for $\mathrm{Re}_h < 1$, the relation (23) gives sufficiently accurate results.

3.2 *Stream function in* MHD *Stokes' flow*

The character of the motion is most concisely expressed in terms of the Stokes' stream function ψ, which is given by

$$(\psi/U_\infty a^2) = -\int_0^\theta r^2(v_r + \cos\theta)\sin\theta\,\mathrm{d}\theta \tag{26}$$

where v_r is the dimensionless radial component of the fluid velocity and is given by

$$v_r = \exp(\mathrm{Re}_h x_1)\frac{\partial\phi_1}{\partial r} + \exp(-\mathrm{Re}_h x_1)\frac{\partial\phi_2}{\partial r} \tag{27}$$

Substituting (27) in (26) and carrying out some mathematical computations, we finally get

$$\frac{\psi}{U_\infty a^2} = -\frac{1}{2}\left(1+\frac{3}{8}\mathrm{Re}_h\right)\left(1-\frac{3}{2r}+\frac{1}{2r^3}\right)r^2\sin^2\theta + 0(\mathrm{Re}_h^2). \tag{28}$$

If we put $\mathrm{Re}_h = 0$, we get Stokes' result of OHD.

4. MHD Oseen's flow

In the Stokes' flow studied in § 3, we neglected the disturbance of the magnetic field and the inertia of the fluid. However, using Oseen type approximation, where quadratic

terms in the disturbance quantities are neglected in the equations of motion and magnetic field, the results of Chester [5] may be improved.

We linearize (2) and (3) by writing

$$\mathbf{H} = \mathbf{h} + \hat{i} \tag{29}$$

where $\mathbf{h}$ is the dimensionless disturbance in the applied magnetic field.

The boundary conditions on $\mathbf{h}$ and $\mathbf{v}$ are

$$\left.\begin{aligned} &r = 1 : h_t \text{ and } \mu_e h_n \text{ be continuous; } \mathbf{v} = -\hat{i} \\ &r = \infty : \mathbf{h} = 0; \ \mathbf{v} = 0 \end{aligned}\right\} \tag{30}$$

where h_t and h_n are, respectively, components of the magnetic field, tangent and normal to the sphere.

From Ampere's law and (29), we conclude that

$$\mathbf{J} \times \mathbf{H} = \frac{1}{\mathrm{Re}_\sigma} (\nabla \times \mathbf{h}) \times (\mathbf{h} + \hat{i}) \tag{31}$$

Substituting (31) in (2) and (29) in (3) and neglecting the small quantities of second order in $\mathbf{v}$ and $\mathbf{h}$, the linearized MHD-Oseen' equations are respectively

$$\mathrm{Re}(\hat{i} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla^2 \mathbf{v} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} (\nabla \times \mathbf{h}) \times \hat{i}, \tag{32}$$

$$\mathrm{Re}_\sigma \nabla \times (\mathbf{v} \times \hat{i}) + \mathrm{Re}_\sigma \nabla \times (\hat{i} \times \mathbf{h}) = \nabla \times (\nabla \times \mathbf{h}) \tag{33}$$

which may be written as

$$\nabla^2 \mathbf{v} - \mathrm{Re} \frac{\partial \mathbf{v}}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} \frac{\partial \mathbf{h}}{\partial x} = \nabla \left(p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} \mathbf{h} \cdot \hat{i} \right), \tag{34}$$

and

$$\nabla^2 \mathbf{h} + \mathrm{Re}_\sigma \left(\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial \mathbf{h}}{\partial x} \right) = 0, \tag{35}$$

where div $\mathbf{h} = 0$ and div $\mathbf{v} = 0$ have been used.

Taking the divergence of (34), we get

$$\nabla^2 \left(p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} \mathbf{h} \cdot \hat{i} \right) = 0 \tag{36}$$

Let,

$$p_0 = p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} \mathbf{h} \cdot \hat{i}, \tag{37}$$

then p_0 satisfies the Laplace equation.

A particular integral of (34) is therefore obtained if we write

$$\left.\begin{aligned} &p_0 = \left(-\mathrm{Re} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma} \right) \frac{\partial \phi}{\partial x}, \\ &\mathbf{v} = \nabla \phi \text{ and } \mathbf{h} = \nabla \phi, \end{aligned}\right\} \tag{38}$$

where ϕ satisfies

$$\nabla^2\phi = 0. \tag{39}$$

For the complete solution we put

$$\mathbf{v} = \nabla\phi + \mathbf{v}_1, \quad \mathbf{h} = \nabla\phi + \mathbf{h}_1, \tag{40}$$

where $\mathbf{v}_1$ and $\mathbf{h}_1$ satisfy the corresponding homogeneous equations

$$\nabla^2\mathbf{v}_1 - \mathrm{Re}\frac{\partial \mathbf{v}_1}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\frac{\partial \mathbf{h}_1}{\partial x} = 0, \tag{41}$$

$$\nabla^2\mathbf{h}_1 + \mathrm{Re}_\sigma\left(\frac{\partial \mathbf{v}_1}{\partial x} - \frac{\partial \mathbf{h}_1}{\partial x}\right) = 0. \tag{42}$$

These equations may be decoupled, if we write

$$\mathbf{v}_1 + \alpha_1\mathbf{h}_1 = (\alpha_2 - \alpha_1)\left[\frac{1}{2k_1}\nabla\chi_1 - \chi_1\hat{i}\right], \tag{43}$$

$$\mathbf{v}_1 + \alpha_2\mathbf{h}_1 = (\alpha_2 - \alpha_1)\left[\frac{1}{2k_2}\nabla\chi_2 - \chi_2\hat{i}\right], \tag{44}$$

where χ_1 and χ_2 satisfy the equations

$$\left(\nabla^2 - 2k_1\frac{\partial}{\partial x}\right)\chi_1 = 0, \quad \left(\nabla^2 - 2k_2\frac{\partial}{\partial x}\right)\chi_2 = 0, \tag{45}$$

$$\alpha_1, \alpha_2 = \frac{1}{2\mathrm{Re}_\sigma}\left[(\mathrm{Re} - \mathrm{Re}_\sigma) \pm \{(\mathrm{Re} - \mathrm{Re}_\sigma)^2 + 4\mathrm{Re}_h^2\}^{1/2}\right], \tag{46}$$

and

$$k_j = (\mathrm{Re} - \mathrm{Re}_\sigma\alpha_j)/2, \tag{47}$$

are the 'modified' Reynolds numbers.

We now define

$$\mathrm{Re}_H = \frac{\mu_e H_\infty^2}{\rho U_\infty^2} = \frac{\mathrm{Re}_h^2}{\mathrm{Re}\,\mathrm{Re}_\sigma} \text{ (Magnetic pressure number).} \tag{48}$$

It may be noted from (47) that for $\mathrm{Re}_H < 1$, each of the 'modified' Reynolds numbers is positive and for $\mathrm{Re}_H > 1$ there is a positive and a negative 'modified' Reynolds number. When $\mathrm{Re}_H = 1$, the 'modified' Reynolds numbers are $(\mathrm{Re} + \mathrm{Re}_\sigma)/2$ and zero. These cases will be considered later in the analysis.

Thus the complete solution is given by

$$\mathbf{v} = \nabla\left(\phi + \frac{\alpha_2}{2k_1}\chi_1 - \frac{\alpha_1}{2k_2}\chi_2\right) - (\alpha_2\chi_1 - \alpha_1\chi_2)\hat{i},$$

$$\mathbf{h} = \nabla\left(\phi - \frac{1}{2k_1}\chi_1 + \frac{1}{2k_2}\chi_2\right) + (\chi_1 - \chi_2)\hat{i},$$

$$p = -\mathrm{Re}\frac{\partial\phi}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\left[\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x} - \frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right) - (\chi_1 - \chi_2)\right], \tag{49}$$

where ϕ, χ_1 and χ_2 are to be obtained from the differential equations (39) and (45) respectively, whose general solutions, in the case of axial symmetry, outside the sphere, which vanish at infinity, are

$$\phi = \sum_{n=0}^{\infty} A_n(\partial/\partial x)^n(1/r), \tag{50}$$

$$\chi_1 = \exp(k_1 x)\sum_{n=0}^{\infty} B_n(\partial/\partial x)^n(\exp(\mp k_1 r)/r), \tag{51}$$

$$\chi_2 = \exp(k_2 x)\sum_{n=0}^{\infty} C_n(\partial/\partial x)^n(\exp(-k_2 r)/r), \tag{52}$$

where the $\mp$ sign in χ_1 correspond to $k_1 \gtrless 0$.

Solutions of the type $\exp[-|k_1|r + k_1 x]/r$ and $\exp[-|k_2|r + k_2 x]/r$ lead to wakes in the directions of increasing $k_1 x$ and $k_2 x$ respectively. As pointed out earlier k_2 is essentially positive so that there is always a downstream wake. On the other hand, $k_1 \gtrless 0$, according as $\mathrm{Re}_H \gtrless 1$. Thus, when the free stream velocity U_∞ is greater than the Alfvén velocity $(H_0/\sqrt{\mu_e/\rho})$ the second wake is also downstream, but when it is less, it is upstream.

Inside the sphere,

$$\mathbf{h} = \nabla\psi, \tag{53}$$

where

$$\psi = \sum_{n=1}^{\infty} D_n r^{2n+1}(\partial/\partial x)^n(1/r). \tag{54}$$

To determine the unknown constants, A_n, B_n, C_n and D_n we apply the boundary conditions (30) on the surface of the sphere $r = 1$ and assume that the sphere has the same magnetic permeability as that of the fluid, we find that

$$A_0 = \frac{3(k_1 - k_2)}{4(\alpha_2 - \alpha_1)k_1 k_2}\left[1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)}\right] \tag{55}$$

$$B_0 = C_0 = \frac{3}{2(\alpha_2 - \alpha_1)}\left[1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)}\right], \tag{56}$$

and

$$A_1 = 1 + (B_0/2)[(\alpha_1 - \alpha_2) + (\pm\alpha_2 k_1 - \alpha_1 k_2)], \tag{57}$$

correct to the first order in k_1 and k_2. The determination of other constants become increasingly complicated.

4.1 *Drag in MHD Oseen' flow*

We now calculate the drag on the sphere. The drag D due to the pressure and viscous forces on the sphere is given by

$$D = -\mu U_0 a \int (lp + m\zeta - n\eta)_{r=1}\, dS, \tag{58}$$

where (l, m, n) are the direction-cosines of the outward drawn normal to the sphere

S: $r = 1$, while η and ζ are the y and z components of curl **v**. The drag due to Maxwell stresses is zero; it is proportional to the flux of **h** through the surface of the sphere.

Keeping (49) in view, the integrand of (58) may be written as

$$l\left[-\mathrm{Re}\frac{\partial\phi}{\partial x}+\frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\left\{\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x}-\frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right)-(\chi_1-\chi_2)\right\}\right]$$
$$+m\left[\alpha_2\frac{\partial\chi_1}{\partial y}-\alpha_1\frac{\partial\chi_2}{\partial y}\right]+n\left[\alpha_2\frac{\partial\chi_1}{\partial z}-\alpha_1\frac{\partial\chi_2}{\partial z}\right], \tag{59}$$

the values are to be taken at $r = 1$.

On the sphere, from (43), (44) and (40) we find

$$\frac{1}{2k}\nabla\chi-\chi\hat{i}=\frac{1}{\alpha_1-\alpha_2}[\hat{i}-\alpha\mathbf{h}+(\alpha+1)\nabla\phi], \tag{60}$$

for $k = k_1$, k_2 etc. This expression, in the right side, differs by a factor $R_M(=\mathrm{Re}_\sigma)$ of reference [7], which is wrongly introduced.

Hence,

$$D=\mu U_\infty a\left(\mathrm{Re}-\frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\right)\int\left(\frac{\partial\phi}{\partial r}\right)_{r=1}\mathrm{d}S$$
$$=-4\pi\mu U_\infty a(\mathrm{Re}-(\mathrm{Re}_h^2/\mathrm{Re}_\sigma))A_0, \tag{61}$$

where A_0 is given by (55). Substituting the values of α_1, α_2, k_1 and k_2 in (55), we find that

$$A_0=-\frac{3}{2\mathrm{Re}(1-\mathrm{Re}_H)}\left[1+\frac{3}{8}K\,\mathrm{Re}\right], \tag{62}$$

$$K=\begin{cases}1 \text{ for } k_1\geqslant 0, \text{ i.e., } \mathrm{Re}_H\leqslant 1\\ \dfrac{2\mathrm{Pr}_m\mathrm{Re}_H+1-\mathrm{Pr}_m}{\{(1-\mathrm{Pr}_m)^2+4\mathrm{Pr}_m\mathrm{Re}_H\}^{1/2}} \text{ for } k_1\leqslant 0, \text{ i.e. } \mathrm{Re}_H\geqslant 1\end{cases} \tag{63}$$

$$\mathrm{Pr}_m=(\mathrm{Re}_\sigma/\mathrm{Re})=\mu_e\sigma v \text{ (Magnetic Prandtl number)} \tag{64}$$

Hence,

$$D=D_S(1+\tfrac{3}{8}K\,\mathrm{Re}) \tag{65}$$

Thus, when $\mathrm{Re}_H\leqslant 1$ i.e. the free stream velocity is greater or equal to the Alfvén velocity the magnetic field has no influence on the drag and it is simply the Oseen' drag of OFD. It is only affected when $\mathrm{Re}_H > 1$. In the purely viscous case $\mathrm{Pr}_m = 0$, $\mathrm{Re}_H = 0$, hence $K = 1$ and we recapture Oseen' drag. On the other hand, when Re and Re_σ are small and are of the same order of magnitude we may take $\mathrm{Pr}_m = 1$ and in that case (63) gives $K = \mathrm{Re}_h/\mathrm{Re}$, which reduces (65) to the MHD Stokes' drag (24).

4.2 *Stream function in MHD Oseen' flow*

In the present case, we have

$$v_r=\frac{\partial\phi}{\partial r}+\frac{\alpha_2}{2k_1}\frac{\partial\chi_1}{\partial r}-\frac{\alpha_1}{2k_2}\frac{\partial\chi_2}{\partial r}-(\alpha_2\chi_1-\alpha_1\chi_2)\cos\theta \tag{66}$$

Substituting (66) in (26), we get
For $k_1 \to 0$

$$\frac{\psi}{U a^2} = A_0(1-\cos\theta) - \frac{A_1}{r}\sin^2\theta + B_0\left(\frac{\alpha_2}{k_1} - \frac{\alpha_1}{k_2}\right)$$
$$\frac{B_0(1+\cos\theta)}{2}\left[\frac{\alpha_2}{k_1}\exp(-k_1 r(1-\cos\theta))\right.$$
$$\left.\frac{\alpha_1}{k_2}\exp(-k_2 r(1-\cos\theta))\right] - \frac{r^2\sin^2\theta}{2} + \ldots \tag{67}$$

For $k_1 \to 0$

$$\frac{\psi}{U a^2} = A_0(1-\cos\theta) - \frac{A_1}{r}\sin^2\theta + \frac{B_0\alpha_2}{2k_1}(1-\cos\theta)\exp(k_1 r(1+\cos\theta))$$
$$+ \frac{B_0\alpha_1}{2k_2}\left[(1+\cos\theta)\exp(-k_2 r(1-\cos\theta)) - 2\right]$$
$$\frac{r^2\sin^2\theta}{2} + \ldots \tag{68}$$

In the non-magnetic case, $k_1 \to 0$, where $\mathrm{Re}_h \to 0$, we have

$$k = k_1 = k_2 = \frac{\mathrm{Re}}{2}$$
$$A_0 = \frac{3}{2\mathrm{Re}}\left(1+\frac{3}{8}\mathrm{Re}\right) \tag{69}$$
$$B_0(\alpha_2 - \alpha_1) = \frac{3}{2}\left(1+\frac{3}{8}\mathrm{Re}\right)$$
$$A_1 = 1 + \frac{3}{4}\left(1+\frac{3}{8}\mathrm{Re}\right)\left(\frac{\mathrm{Re}}{2}-1\right)$$

Therefore, the stream function (67) reduces to

$$\frac{\psi}{U a^2} = \frac{3}{2\mathrm{Re}}\left(1+\frac{3}{8}\mathrm{Re}\right)(1+\cos\theta)[1-\exp(-kr(1-\cos\theta))]$$
$$-\left(\frac{1}{4}+\frac{3}{32}\mathrm{Re}+\frac{9}{64}\mathrm{Re}^2\right)\frac{\sin^2\theta}{r} - \frac{r^2\sin^2\theta}{2}. \tag{70}$$

For $\mathrm{Re} \ll 1$, it is the same as given by Milne-Thomson [10] in OFD.

5. Conclusion

In MHD Stokes' flow there is a wake downstream unlike OFD Stokes' flow for $\mathrm{Re} \ll 1$, which modifies the OFD Stokes' drag expression in a similar manner as by OFD Oseen' flow to the Stokes' flow with Re replaced by Re_h ($\mathrm{Re}_h \ll 1$). In MHD Oseen's flow, when the free stream velocity is greater or equal to Alfvén velocity the formation of the wake

is only downstream and the magnetic field has no influence on the Oseen drag of OFD. On the other hand, when the free stream velocity is less than the Alfvén velocity we have both a wake upstream and a wake downstream which modifies the OFD Oseen's drag expression and increases the drag due to the presence of the magnetic field.

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Weyl fractional calculus and Laplace transform

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Abstract. The Weyl fractional calculus is developed to obtain Laplace transforms of $t^q\phi(t)$ (for all real values of q) where $\phi(t)$ is taken in the form of $f(a\sqrt{(t^2-b^2)})$ and certain other forms. Also, a generating function involving H-function of several variables is established with the help of generalized Taylor series.

Keywords. Weyl fractional calculus; Laplace transform; H-function.

1. Introduction

Let A denote a class of good functions. By a good function f, we mean a function which is everywhere differentiable any number of times and if it and all of its derivatives are $O(x^{-\nu})$, for all ν as x increases without limit (see [4]).

We define the Weyl fractional derivative of a function $\phi(z)$ as follows: Let $\phi \in A$, then

$$ {}_zD_\infty^q \phi(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1} \phi(u)\, du,\ q < 0, \tag{1}$$

and

$$ {}_zD_\infty^q \phi(z) = \frac{d^r}{dz^r} ({}_zD_\infty^{q-r} \phi(z)),\ q \geqslant 0, \tag{2}$$

r being a positive integer such that $r > q$.

We recall the definition of Laplace transform of $f(t)$ in the form:

$$ \psi(p) = L[f(t); p] = \int_0^\infty \exp(-pt) f(t)\, dt,\ p > 0. \tag{3}$$

Recently the Laplace transform of $t^q f(t)$, for arbitrary q (real), was obtained in [5] in the theory of Weyl fractional calculus. In this paper, we further develop the Laplace transform of $t^q\phi(t)$ in this theory of Weyl fractional calculus, where $\phi(t)$ is taken in terms of $f(a\sqrt{(t^2-b^2)})$, $f^{(n)}(a\sqrt{(t^2-b^2)})$, $f^{(n)}(t)$ etc., and $f^{(n)}$ denotes the nth derivative of f. As pointed out in ([3], p. 16) the importance of the function $f(a\sqrt{(t^2-b^2)})$ lies in the fact that in certain technical problems involving wave motion, *e.g.* the theory of compressional shock waves, of uniform transmission lines, and that of exponential loudspeaker horns, the variable has this form, b being of the same dimension as t.

We define the Laplace transform of the function $f(a\sqrt{(t^2-b^2)})$ to be that of

$$ \left.\begin{aligned} F(t) &= f(a\sqrt{(t^2-b^2)}) \quad \text{when } t > b > 0 \\ &= 0 \qquad\qquad\qquad\ \ \text{when } t < b \end{aligned}\right\} \tag{4}$$

so that we have

$$\phi(p) = L[F(t); p] = \int_b^\infty \exp(-pt) f(a\sqrt{(t^2-b^2)})\,\mathrm{d}t,\ p > 0. \tag{5}$$

For the definition of the H-function of several variables *viz.*

$$H[x_1, \ldots, x_r] = H^{0,l:(m',n'),\ldots,(m^{(r)},n^{(r)})}_{A,C:[B',D'],\ldots,[B^{(r)},D^{(r)}]}[x_1, \ldots, x_r] \tag{6}$$

and its conditions of convergence etc., we refer to [6]. The appearance of asterisks in (19) and (20) below indicates that the parameters at those places are the same as in the H-function on the right side of (6) at the corresponding places.

2. Main results

THEOREM 1.

If $\phi(p)$ is given by (5), then, for all real q,

$$\text{(i)}\ (-1)^q\, {}_pD^q_\infty \{\phi(p)\} = L[t^q f(a\sqrt{(t^2-b^2)}); p]; \tag{7}$$

$$\text{(ii)}\ (-1)^q\, {}_pD^q_\infty (p^n \phi(p)) = L[t^q f^{(n)}(a\sqrt{(t^2-b^2)}); p], \tag{8}$$

where $f^{(n)}$ denotes the nth derivative of $f(a\sqrt{(t^2-b^2)})$ with respect to t, provided that $f^{(r)}(0) = 0$, $[r = 0, 1, \ldots, (n-1)]$;

$$\text{(iii)}\ (-1)^q\, {}_pD^q_\infty (p^{-n}\phi(p)) = L[t^q K_n(a\sqrt{(t^2-b^2)}); p], \tag{9}$$

where

$$K_n(a\sqrt{(t^2-b^2)}) = \int_0^t \ldots \int_0^t f(a\sqrt{(t^2-b^2)}) \cdot (\mathrm{d}t)^n, \tag{10}$$

provided that $\lim \exp(-pt)\, K_1^{(r)}(a\sqrt{(t^2-b^2)}) = \lim \exp(-pt) \int_0^t \{[\int_b^\tau \mathrm{d}\tau]^r\}$

$$\times f(a\sqrt{(\tau^2-b^2)})\,\mathrm{d}\tau \to 0$$

as $t \to \infty$, for $r = 0, 1, \ldots, (n-1)$.

THEOREM 2.

If $\psi(p)$ is given by (3), then, for all real q,

$$\text{(i)}\ (-1)^q\, {}_pD^q_\infty (p^n \psi(p)) = L[t^q f^{(n)}(t); p], \tag{11}$$

provided that $f^{(r)}(0) = 0$, $[r = 0, 1, \ldots, (n-1)]$;

$$\text{(ii)}\ (-1)^q\, {}_pD^q_\infty [p^{-n}\psi(p)] = L[t^q F_n(t); p], \tag{12}$$

where

$$F_n(t) = \int_0^t \ldots \int_0^t f(t)\,(\mathrm{d}t)^n, \tag{13}$$

provided that $\lim \exp(-pt) F_1^{(r)}(t) = \lim \exp(-pt) \int_0^t \{[\int_0^\tau \mathrm{d}\tau]^r f(\tau)\}\,\mathrm{d}\tau \to 0$ as $t \to \infty$, for $r = 0, 1, \ldots, (n-1)$.

Proofs: To prove (7), we have, in view of (1) and (5), for $q < 0$,

$$(-1)^q\, {}_pD^q_\infty \phi(p) = \frac{1}{\Gamma(-q)} \int_p^\infty (u-p)^{-q-1}\, \phi(u)\,\mathrm{d}u$$

$$= \frac{1}{\Gamma(-q)} \int_p^\infty (u-p)^{-q-1} \times \left(\int_b^\infty f(a\sqrt{(t^2-b^2)}) \exp(-ut)\,\mathrm{d}t \right) \mathrm{d}u.$$

We now change the order of integration, use ([2], p. 202, (11)) to evaluate the inner integral on the right side to get

$$(-1)^q {}_pD_t^q \phi(p) = L[t^q f(a\sqrt{(t^2-b^2)}); p], \text{ for } q < 0.$$

For $q > 0$, invoking the definition (2), we can write

$$(-1)^q {}_pD_t^q \phi(p) = \frac{d^r}{dp^r}\left(\int_b^t \exp(-pt) t^{q-r} f(a\sqrt{(t^2-b^2)})\,dt\right), \; r > q.$$

Differentiating r-times under the sign of integration ([3], p. 172), we again find that

$$(-1)^q {}_pD_t^q \phi(p) = L[t^q f(a\sqrt{(t^2-b^2)}); p], \text{ for } q \geq 0.$$

This completes the proof of (7).

The results (8), (9), (11) and (12) are similarly established by using ([3], p. 191, equation (22); p. 24, Theorem 5(a); p. 21, equation (9); p. 24, equation (5)).

If we take $q = -1$ and $f(t) = J_0(t)$ in (7), use ([1], p. 191, equation (9)) so that

$$\phi(p) = R^{-1}\exp(-bR),\; R = \sqrt{(p^2+a^2)} \tag{14}$$

then, we get, on using ([1], p. 191, equation (11)) and (14)

$$ {}_pD_t^{-1}[\exp(-bR)/R] = \exp(-bR)\int_0^t \exp(-u)$$
$$\times (u^2+2buR)^{-1/2}\,du,\; p > a, \tag{15}$$

where $R = \sqrt{(p^2+a^2)}$.

Remark: The results given in theorems 1 and 2 can further be generalised in view of the following relation, which can be easily established:

If $\psi(p)$ is given by (3), then, for all q,

$$(-1)^q {}_pD_t^q [\exp(-ap)\psi(p)] = L[t^q f(t-a); p]. \tag{16}$$

3. The generating function

The generalized Taylor's formula in Weyl fractional calculus is given by ([5], p. 189, equation (10)):

$$g(p+t) = \sum_n \frac{at^{an+\eta}}{\Gamma(an+\eta+1)} {}_pD_t^{an+\eta}[g(p)], \tag{17}$$

valid for all t on the circle $|t-p| = 1$, provided that η is an arbitrary complex number, a is such that $0 < a < 1$ and $g(p) \in A$.

Let

$$f(t) = t^{\alpha-1}\exp(-bt)\,H[a_1 t^{h_1}, \ldots, a_r t^{h_r}],$$
$$h_i > 0,\; i = 1, \ldots, r,\; b \geq 0, \tag{18}$$

then

$$\text{(i)}\; g(p) = L[f(t); p] = (p+b)^{-\alpha} H^{0,l+1:*}_{A+1,C:*}\left[\begin{matrix} S:*; \\ *:*; \end{matrix}\right.$$
$$\left. a_1(p+b)^{-h_1}, \ldots, a_r(p+b)^{-h_r}\right] \tag{19}$$

where

$$S:[1-\sigma:h_1,\ldots,h_r],[(a):\alpha',\ldots,\alpha^{(r)}]$$

provided that $(p+b)>0$, $\mathrm{Re}\left(\sigma+\sum_{i=1}^{r}h_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ $(j=1,\ldots,m^{(i)})$, and

$$\text{(ii)}\ {}_pD_{\infty}^{q}\left(p^{\lambda}H[a_1p^{-h_1},\ldots,a_rp^{-h_r}]\right)$$

$$=(-1)^q\,p^{\lambda-q}\,H_{A+1,C+1:*}^{0,l+1:*}\left[\begin{matrix}[1+\lambda-q:h_1,\ldots,h_r],\\ [1+\lambda:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\alpha',\ldots,\alpha^{(r)}]:*;\\ [(b):\beta',\ldots,\beta^{(r)}]:*;\end{matrix}\ a_1p^{-h_1},\ldots,a_rp^{-h_r}\right] \tag{20}$$

provided $h_i>0$,

$$\mathrm{Re}\left(-\lambda+\sum_{i=1}^{r}h_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\ (j=1,\ldots,m^{(i)}).$$

(19) and (20) can be easily established.

The following generating function is now obtained on substituting for $g(p)$ from (19) into (17) and using (20) and making slight changes in the parameters therein:

$$(p+t)^{-\sigma}\,H_{A+1,C+1:*}^{0,l+1:*}\left[\begin{matrix}[1-\sigma:h_1,\ldots,h_r],[(a):\alpha',\ldots,\alpha^{(r)}]:*;\\ *\qquad\qquad\qquad\qquad\qquad\qquad\quad:*;\end{matrix}\right.$$

$$\left.a_1(p+t)^{-h_1},\ldots,a_r(p+t)^{-h_r}\right]$$

$$=p^{-\sigma}\sum_{n=-\infty}^{\infty}\frac{a(-t/p)^{an+\eta}}{\Gamma(an+\eta+1)}\times$$

$$\times\ H_{A+1,C:*}^{0,l+1:*}\left[\begin{matrix}[1-\sigma-an-\eta:h_1,\ldots,h_r],[(a):\alpha',\ldots,\alpha^{(r)}]:*;\\ *\qquad\qquad\qquad\qquad\qquad\qquad\quad:*;\end{matrix}\right.$$

$$\left.a_1p^{-h_1},\ldots,a_rp^{-h_r}\right], \tag{21}$$

valid under conditions mentioned with (17).

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A maximum flow problem with intermediate node requirements

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Abstract. The conventional Maximum flow problem is modified to take account of possible requirements at intermediate nodes across which flow takes place. This is achieved by incorporating pseudo or priority arcs to act as thresholds controlling out-flow from the nodes and modifying the Ford and Fulkerson algorithm to take account of these thresholds.

Effect of introducing these threshold–requirements at intermediate nodes on the final flow into the sink in the network is examined by some numerical examples.

Keywords. Maximum flow; intermediate node; threshold requirements; pseudo or priority arcs; conservation of flows.

1. Introduction

The Network flow optimization problem considered here is that of determining a maximal steady state flow from a node designated as "the source" to a node designated as "the sink" in a network under the following situation:

(a) Arc capacities are non-negative,

(b) Intermediate nodes have finite requirements.

However, the requirement at a node without a positive in-flow need not be satisfied while out-flow from a node with positive in-flow is allowed only if in-flow into the node exceeds its requirement.

2. Some definitions and notations

A Network $N(V; \mathscr{A} \subseteq (V \times V); C)$ is a topological structure having a non-empty set V of elements called 'nodes' (vertices, points) and a set $\mathscr{A}$ of elements called 'arcs' (lines, branches, edges). The set $\mathscr{A}$ is a proper sub-set of elemets from the set $(V \times V)$, i.e., elements of $\mathscr{A}$ are ordered pairs of elements from V and are called 'directed arcs'. Every arc in the network has a positive number associated with it; this number is called the capacity of the arc.

Nodes are represented by small circles with the corresponding node names inside this. Nodes are named as $1, 2, \ldots, n$. Node 1 is the source node and node n is the sink node.

$$V = \{v : v = 1, 2, \ldots, n\} \text{ is the set of nodes.}$$

$$\mathscr{A} = \{a_p : p = 1, 2, \ldots, L;\ a_p = (i, j),\ i, j \in V \text{ and } i \neq j\}$$

is the set of ordered pairs of elements from V forming the set of arcs. L is a positive

integer not more than $(n^2 - n)$.

$K = \{C_p\colon\ p = 1, 2, \ldots, L\}$ is the set of capacities of the arcs a_p, $p = 1, 2, \ldots, L$ respectively.

The intermediate nodes j for $j = 2, 3, \ldots, (n-1)$ have the requirements $R_j > 0$. For convenience, we define $R_1 = R_n = 0$.

It is more convenient to give the capacities of the arcs (i, j) by the corresponding matrix entries C_{ij} of an $(n \times n)$ matrix; for "non-existing arcs" i.e., ordered pairs $(i, j) \notin \mathscr{A}$, C_{ij} is defined as zero. Similarly, $A = (a_{ij})$ is defined as the connectivity matrix of the network, where

$$a_{ij} = \begin{cases} 1 \text{ if } (i,j) \in \mathscr{A} \\ 0 \text{ if } (i,j) \notin \mathscr{A} \end{cases}$$

3. Maximum flow problem (MFP for short)

The conventional maximum flow problem (MFP), (Ford and Fulkerson [1], Hu [2]) can be stated as follows:

Find real numbers $\{f_{ij}\}$, such that

$$0 \leqslant f_{ij} \leqslant C_{ij}; \tag{1}$$

$$\sum_{i=1}^{n} f_{ij} - \sum_{k=1}^{n} f_{jk} = 0 \quad \text{if} \quad j \neq 1 \text{ and } n \tag{2}$$

$$\sum_{i=1}^{n} f_{ij} - \sum_{k=1}^{n} f_{jk} = T \quad \text{if} \quad j = n; \tag{3}$$

$$\sum_{i=1}^{n} f_{ij} - \sum_{k=1}^{n} f_{jk} = -T \quad \text{if} \quad j = 1 \tag{4}$$

which also maximise $T = \sum_{i=1}^{n} f_{in}$

Equation (2) is called 'Law of Conservation of Flow' or 'Material balance equation' or 'Kirchhoff's Law for Flows', and indicates that 'No material is lost or added during transit'. In the present problem, (2) is relaxed as follows:

Every intermediate node j, with a positive inflow will have a positive outflow, only after meeting its requirement. However, the requirement of a node without positive inflow is ignored.

4. Maximum flow problem with requirements (MFPR)

Find a set of real numbers $\{f_{ij}\}$ to maximise $T_1 = \sum_{i=1}^{n} f_{in}$ such that

$$0 \leqslant f_{ij} \leqslant C_{ij} \tag{5}$$

Let $I_j = \sum_{i=1}^{n} f_{ij}$ be the inflow for j, then

If $\quad I_j \leqslant R_j, \quad f_{jk} = 0$ for all k while, (6)

$$I_j > R_j, \quad \text{then } I_j = R_j + \sum_{k=1}^{n} f_{jk} \tag{7}$$

In what follows an algorithm for solving MFPR is presented. The algorithm is got by uitably modifying the algorithm due to Ford and Fulkerson for MFP.

. Modified algorithm for MFPR

(i) Read n; $C_{ij}((i = 1, 2, \ldots, n), (j = 1, 2, \ldots, n))$; $R_i(i = 1, 2, \ldots, n)$
(ii) Do step 3 varying i from 1 to n
(iii) $\mathrm{ST}_i \leftarrow 0$; $\mathrm{FL}_i \leftarrow 0$; $\mathrm{AF}_i \leftarrow 0$; $\mathrm{NOD}_i \leftarrow 0$
(iv) Code $\leftarrow 0$; $\mathrm{NOD}_1 \leftarrow n+1$
(v) Do step 6 varying i from 1 to n
(vi) If $C_{1i} \neq 0$; $\mathrm{NOD}_i \leftarrow 1$; $\mathrm{ST}_i \leftarrow 1$; Code $\leftarrow 1$ else go to 5
(vii) If Code $= 0$ go to 36
(viii) If $\mathrm{NOD}_n \neq 0$ go to 20
(ix) $K \leftarrow 1$
(x) Code $\leftarrow 0$
(xi) Do step 12 through 15 varying i from 1 to n
(xii) If $(\mathrm{ST}_i - K) \neq 0$ go to 11
(xiii) Do step 14 through 15 varying j from 1 to n
(xiv) If $C_{ij} \neq 0$; code $\leftarrow 1$ else go to 13
(xv) If $\mathrm{NOD}_j = 0$; $\mathrm{NOD}_j \leftarrow 1$; $\mathrm{ST}_j \leftarrow (K+1)$ else go to 13
(xvi) If Code $= 0$ go to 36
(xvii) If $\mathrm{NOD}_n \neq 0$ go to 23
(xviii) $K \leftarrow K+1$
(xix) If $K \geqslant n$ go to 36 else go to 10
(xx) $C_{n1} = C_{1n}$; $C_{1n} = 0$
(xxi) Print n, C_{n1} (Prints out the direct path and amount of flow between 1 and n)
(xxii) Go to 2
(xxiii) $K \leftarrow K+1$; $M \leftarrow K+1$
(xxiv) $\mathrm{IJ}_M \leftarrow n$
(xxv) Do step 26 varying i from K to 1
(xxvi) $L1 \leftarrow \mathrm{IJ}_{i+1}$; $\mathrm{IJ}_i \leftarrow \mathrm{NOD}_{L1}$
(xxvii) $N2 \leftarrow K+1$
(xxviii) Do step 29 varying i from 1 to K
(xxix) $\mathrm{FL}_i \leftarrow C_{\mathrm{IJ}_i \mathrm{IJ}_{j+1}}$; $\mathrm{IJR}_i \leftarrow R_{\mathrm{IJ}_i}$
(xxx) Calculate flow and store the flow in AFL and store the remaining requirements to be met in IJR (procedure is given below)
(xxxi) Do step 32 through 33 varying i from 1 to $(N2-1)$
(xxxii) $\mathrm{I2} \leftarrow \mathrm{IJ}_i$; $\mathrm{I3} \leftarrow \mathrm{IJ}_{i+1}$
(xxxiii) $C_{I2\,I3} \leftarrow C_{I2\,I3} - \mathrm{AFL}_i$; $C_{I3\,I2} \leftarrow C_{I3\,I2} + \mathrm{AFL}_i$; $R_{I2} \leftarrow \mathrm{IJR}_i$
(xxxiv) Print IJ; FL; AFL
(xxxv) Go to 2
(xxxvi) Print "Path search is over"
(xxxvii) End

6. Calculation of AFL and IJR

To calculate AFL and IJR the following procedure is adopted. AFL consists of flow values in different arcs included in the path. FL contains the remaining capacities of the arcs in the path. $\{F_{jn}\}$ contains the flow values in the "Priority arcs" for nodes 2 to $(n-1)$.

(i) Do step 2 through 7 varying i from 1 to $(n-2)$
(ii) $K \leftarrow$ Minimum of $\{FL_1, FL_2, \ldots, FL_i, IJR_{i+1}\}$
(iii) Do step 4 through 5 varying j from 1 to i
(iv) $FL_j \leftarrow FL_j - K$
(v) $AFL_j \leftarrow AFL_j + K$
(vi) $IJR_{i+1} \leftarrow IJR_{i+1} - K$
(vii) $F_{(i+1)n} \leftarrow F_{(i+1)n} + K$
(viii) $K1 \leftarrow$ minimum of $\{FL_1, \ldots, FL_{n-1}\}$
(ix) Do step 10 through 11 varying i from 1 to $(n-1)$
(x) $FL_i \leftarrow FL_i - K1$
(xi) $AFL_i \leftarrow AFL_i + K1$
(xii) Return

7. Numerical illustration

A numerical example illustrating the working of the algorithm for the MFPR follows

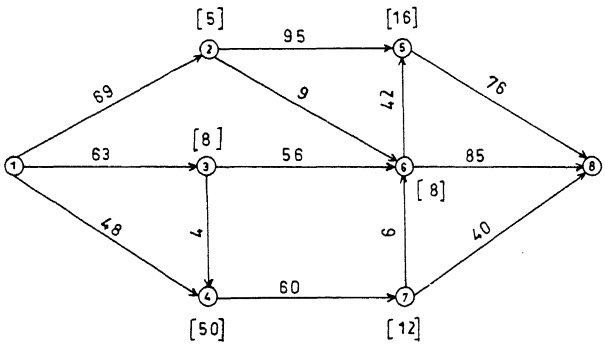


Figure 1.

Figure 1 is a schematic repesentation of a network with capacitated arcs and node requirements. For this network we have,

$$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$\mathscr{A} = \{(1, 2), (1, 3), (1, 4), (2, 5), (2, 6), (3, 4), (3, 6), (4, 7), (5, 8), (6, 5), (6, 8), (7, 6), (7, 8)\}$$

$$K = \{69, 63, 48, 95, 9, 4, 56, 60, 76, 42, 85, 6, 40\}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 69 & 63 & 48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 95 & 9 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 56 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 76 \\ 0 & 0 & 0 & 0 & 42 & 0 & 0 & 85 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = \{0, 5, 8, 50, 16, 8, 12, 0\}$

Numbers present on the arcs are "Capacities". Numbers besides the nodes (represented by number inside the circle) in square brackets are requirements of the nodes.

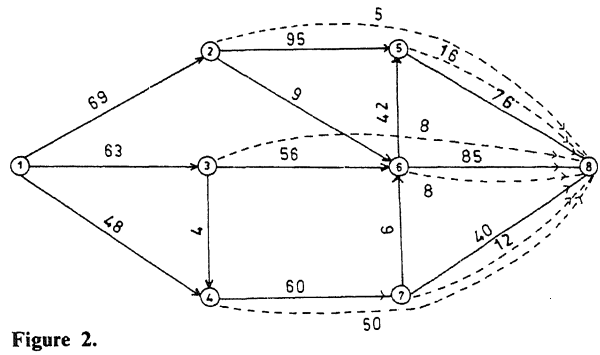


Figure 2.

Figure 2 is obtained from figure 1 by suitable modifications as per the algorithm. Pseudo arcs i.e., Priority arcs are represented by dotted lines. Numbers on the priority arcs are requirements of the corresponding intermediate nodes. The flow matrix (f_{ij}) is augmented by a column (F_{jn}) of the flows corresponding to the flows in priority arcs.

Requirement R_j of node j is sent directly to the sink through a pseudo arc called a 'Priority arc'; capacity of this priority arc is equal to R_j. This arc should be satisfied fully before a positive outflow from node j can occur. In other words, inflow into node j must be greater than R_j in order to have a positive outflow from node j, i.e., $\sum_{i=1}^{n} f_{ij} > R_j$ if, in a flow pattern (f_{ij}), $\sum_{k=1}^{n} f_{jk} > 0$. The flow in priority arc $F_{jn} = \min \left(\sum_{i=1}^{n} f_{ij}, R_j \right)$. The flows (f_{jk}) can be positive only if $F_{jn} = R_j$.

From figure 2 it is clear that the pseudo arcs named priority arcs are different from 'Multiple arcs', where multiple arcs are defined as 'More than one arc exists between the given pair of nodes (i.e., parallel arcs)'.

The vectors {NOD} and {ST} are used to identify a path from source to sink during the different stages of executing the algorithm. At a given stage, if $C_{ij} > 0$, then NOD_j is modified to the values i, ST_j is modified to the value $(\text{ST}_i + 1)$, indicating the step number in which node j is reached. Code is a check counter which takes values 0 or 1 to check the existence of a path.

The capacity matrix C given in §7 is augmented by a row R of requirements and a

column F for 'Flow in priority arcs'. Initially F is set to zero. Augmentations are shown in table 1. Now the iteration starts.

ITERATION 1

Table 1.

		1	2	3	4	5	6	7	8	F	NOD	ST
			(−)									
	1	0	69	63	48	0	0	0	0	0	9	0
		(+)				(−)				(+)		
	2	0	0	0	0	95	9	0	0	0	1	1
	3	0	0	0	4	0	56	0	0	0	1	1
	4	0	0	0	0	0	0	60	0	0	1	1
C =			(+)						(−)	(+)		
	5	0	0	0	0	0	0	0	76	0	2	2
	6	0	0	0	0	42	0	0	85	0	2	2
	7	0	0	0	0	0	6	0	40	0	4	2
						(+)						
	8	0	0	0	0	0	0	0	0	0	5	3
			(−)			(−)						
	R	0	5	8	50	16	8	12	0			

Initially $\{NOD_i\}$ and $\{ST_i\}$ are set to 'zeroes' for all i. The vectors $\{NOD\}$ and $\{ST\}$ are modified as per the algorithm.

$$\text{Code} = 0;\ NOD_1 = 9$$

$$C_{12} \neq 0;\ NOD_2 = 1;\ ST_2 = 1;\ \text{Code} = 1$$

$$C_{13} \neq 0;\ NOD_3 = 1;\ ST_3 = 1;\ \text{Code} = 1$$

$$C_{14} \neq 0;\ NOD_4 = 1;\ ST_4 = 1;\ \text{Code} = 1$$

Since Code = 1 and $NOD_n = 0$; $K = 1$; Code = 0 $(ST_2 - K) = 0$. For a positive C_{ij} in matrix C with $i = 2$,

$$C_{25} \neq 0;\ NOD_5 = 2;\ ST_5 = 2;\ \text{Code} = 1$$

$$C_{26} \neq 0;\ NOD_6 = 2;\ ST_6 = 2;\ \text{Code} = 1$$

Proceeding in the same way, vectors $\{NOD\}$ and $\{ST\}$ are modified. Results are as shown in table 1.

To identify path, we proceed in vector $\{NOD\}$. $NOD_8 = 5$ with $ST_8 = 3$. Hence node 8 is reached directly from node 5 and is reached in 3 steps from the source.

$NOD_5 = 2$ with $ST_5 = 2$. Hence node 5 is reached directly from node 2 and is reached from source in 2 steps.

$NOD_2 = 1$ with $ST_2 = 1$. Hence node 2 is reached directly from node 1 and is reached from source in 1 step.

Thus we have the path $1 \rightarrow 2 \rightarrow 5 \rightarrow 8$.

Now, we construct table 2 below, where the serial number can go up to 8.

Table 2.

Sl. No.	1	2	3	4
Path	1	2	5	8
FL	69	95	76	
IJR	0	5	16	0

FL_i = Capacity of the arc $(i, i+1)$ in the path.

IJR_i = Requirement of the node i in the path.

Table 3.

FL	0	31	28
IJR	0	0	0
AFL	69	64	48
F	0	5	16

Table 3: constructed from table 2 according to the §6 of the algorithm.

Super-impose the values from table 3 to table 1 such that the corresponding values of AFL are super-imposed for a + ve sign and FL values are super-imposed for a − ve sign of the matrix '*C*'. Similarly, IJR is super-imposed into {R} for a − ve sign and *F* is super-imposed into {F} for a + ve sign. By super-imposition we mean that the corresponding values are added for a + ve sign and substracted for a − ve sign to the existing values. The resultant matrix is shown in table 4. Now, the second iteration starts.

ITERATION 2

Table 4.

		1	2	3	4	5	6	7	8	*F*	NOD	ST
				(−)								
	1	0	0	63	48	0	0	0	0	0	9	0
	2	69	0	0	0	31	9	0	0	5	0	0
		(+)					(−)			(+)		
$C =$	3	0	0	0	4	0	56	0	0	0	1	1
	4	0	0	0	0	0	0	60	0	0	1	1
	5	0	64	0	0	0	0	0	28	16	6	3
				(+)					(−)	(+)		
	6	0	0	0	0	42	0	0	85	0	3	2
	7	0	0	0	0	0	6	0	40	0	4	2
							(+)					
	8	0	0	0	0	48	0	0	0	0	6	3
				(−)			(−)					
	R	0	0	8	50	0	8	12	0			

We have the path $1 \rightarrow 3 \rightarrow 6 \rightarrow 8$.

Now, we construct the table 5 according to the path and the corresponding capacities alongwith requirements.

Table 5.

Sl. No.	1	2	3	4
Path	1	3	6	8
FL	63	56	85	
IJR	0	8	8	0

Table 6 is obtained from table 5 as per "Calculation of AFL and IJR" of the algorithm.

Table 6.

FL	0	1	38
IJR	0	0	0
AFL	63	55	47
F	0	8	8

The values of table 6 are once again super-imposed on table 4 as explained in iteration 1. Remaining capacities and requirements after the necessary changes are given in table 7. Now, we start with iteration 3.

ITERATION 3

Table 7.

$C =$

	1	2	3	4	5	6	7	8	*F*	NOD	ST
				(−)							
1	0	0	0	48	0	0	0	0	0	9	0
2	69	0	0	0	31	9	0	0	5	0	0
3	63	0	0	4	0	1	0	0	8	0	0
	(+)						(−)		(+)		
4	0	0	0	0	0	0	60	0	0	1	1
5	0	64	0	0	0	0	0	28	16	0	0
6	0	0	55	0	42	0	0	38	8	7	3
				(+)				(−)	(+)		
7	0	0	0	0	0	6	0	40	0	4	2
							(+)				
8	0	0	0	0	48	47	0	0	0	7	3
				(−)			(−)				
R	0	0	0	50	0	0	12	0			

We have the path $1 \rightarrow 4 \rightarrow 7 \rightarrow 8$.

Tables 8 and 9 are obtained as per the algorithm already explained in iterations 1 and 2.

Table 8.

Sl. No.	1	2	3	4
Path	1	4	7	8
FL	48	60	40	
IJR	0	50	12	0

Table 9.

FL	0	60	40
IJR	0	2	12
AFL	48	0	0
F	0	48	0

Similar to the earlier operations of superimpositions, the necessary changes of table 7 are made to obtain table 10 and we proceed further to iteration 4.

ITERATION 4

Table 10.

$C =$

	1	2	3	4	5	6	7	8	*F*	NOD	ST
1	0	0	0	0	0	0	0	0	0	9	0
2	69	0	0	0	31	9	0	0	5	0	0
3	63	0	0	4	0	1	0	0	8	0	0
4	48	0	0	0	0	0	60	0	48	0	0
5	0	64	0	0	0	0	0	28	16	0	0
6	0	0	55	0	42	0	0	38	8	0	0
7	0	0	0	0	0	6	0	40	0	0	0
8	0	0	0	0	48	47	0	0	0	0	0
R	0	0	0	2	0	0	12	0			

From table 10, we find that no feasible path from source (i.e., node 1) to sink (i.e. node 8) exists. Hence, the algorithm is terminated.

It is worth noting the following points of contrast with the usual MFP.

(i) Inflow into node 4, i.e., $\sum_{i=1}^{8} f_{i4} = 48$ is not greater than its requirement $R_4 = 50$. Hence, no outflow takes place from node 4 i.e., $f_{4j} = 0$ for all j.

(ii) Total flow into the sink $= T_1 = 48 + 47 = 95$. However, total outflow from the source is $69 + 63 + 48 = 180 > 95$.

(iii) Requirement of the node 4 is satisfied partially whereas that of node 7 is not met at all. However, flow-out from these nodes is zero.

Table 11.

Capacity matrix	A				B				C				D			
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*
1.	178	178	0	0(4)*	205	170	35	35(5)	205	170	35	35(6)	205	178	27	35(7)
2.	158	158	0	0(12)	158	133	25	25(13)	158	133	25	25(14)	158	150	8	25(15)
3.	158	158	0	0(16)	158	97	61	61(17)	158	97	61	61(18)	158	151	7	61(19)
4.	158	158	0	0(20)	158	137	21	70(21)	158	123	35	70(22)	158	138	20	70(23)
5.	158	158	0	0(24)	158	117	41	70(25)	158	103	55	70(26)	158	155	3	70(27)
6.	178	178	0	0(28)	225	160	65	65(29)	225	160	65	65(30)	225	178	47	65(31)
7.	178	178	0	0(32)	241	70	171	212(33)	241	70	171	212(34)	241	125	116	212(35)
8.	114	114	0	0(36)	121	102	19	49(37)	121	102	19	49(38)	121	114	7	49(39)
9.	133	133	0	0(40)	152	121	31	31(41)	152	121	31	31(42)	152	133	19	31(43)
10.	157	157	0	0(44)	179	148	31	31(45)	179	148	31	31(46)	179	157	22	31(47)
11.	150	150	0	0(48)	152	121	31	31(49)	152	121	31	31(50)	152	145	7	31(51)
12.	104	104	0	0(52)	104	73	31	31(53)	104	73	31	31(54)	104	97	7	31(55)

A: Solution by conventional MFP; *B*: Solution by MFPR [i.e., as per case (i)]; *C*: Solution as per case (ii); *D*: Solution as per case (iii).

a: Total out-flow from the source $\left(\sum_{i=1}^{n} f_{1i}\right)$; *b*: Total in-flow into the sink $\left(\sum_{i=1}^{n} f_{in}\right)$; *c*: Total requirement satisfied $\left(\sum_{i=1}^{n} r_i\right)$; *d*: Total requirement $\left(\sum_{i=1}^{n} R_i\right)$.

*Figure in bracket indicates the corresponding figure numbers.

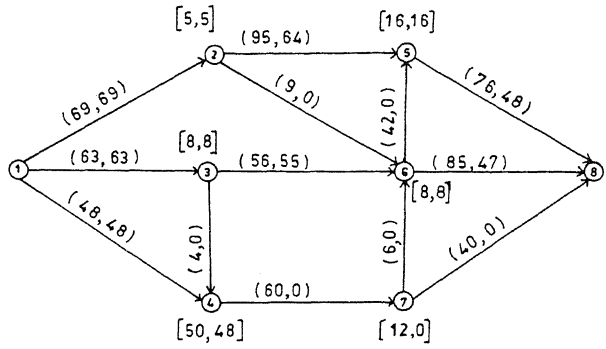


Figure 3.

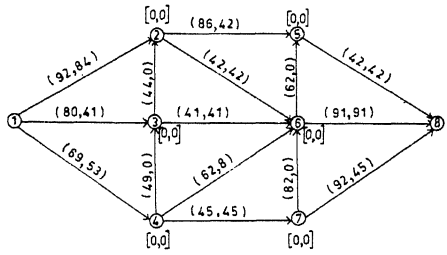


Figure 4.

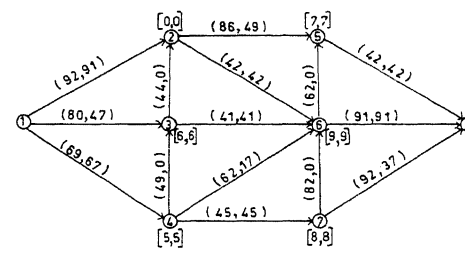


Figure 5.

Figure 3 gives the final picture of the flow values in arcs and the requirements. The numbers besides the arcs are (capacity, flow) and besides the nodes are [Requirement, Requirement satisfied] respectively.

With all the intermediate node requirements zero, we have the conventional MFP, for which we get a maximum flow (MF). By introducing the threshold requirements, we have MFPR and we get MF by the algorithm in §§ 5 and 6.

In general, the MF in MFP is not same as the MF in MFPR. However, introduction of the requirements may or may not necessarily make the situation worse. Thus, for example, we can see from figure 4, with requirements vector 'zero', the MF is 178 units whereas from figure 5, we observe that the MF is 170 units only, with the requirements vector $R = [0, 0, 6, 5, 7, 9, 8, 0]$. Here, we have the total requirement 35 units, satisfied completely. From figure 8, it is also seen that by selecting the requirements vector properly, we can have the MF equal to 178 units same as the MF in figure 4 and moreover, we can satisfy more of the requirements than in figure 5. In this network, we can satisfy 63 units of the requirement from the requirement vector $R = [0, 8, 39, 7, 0, 9, 0, 0]$.

A trivial way of introducing the requirements vecor without upsetting the MF is to have the requirements only at nodes directly connected to the source, their values being the residual capacities, $(C_{1j} - f_{1j})$ of the corresponding arcs. However, this simplicity disappears when requirement is introduced at a node not directly connected with the source.

This problem of determining a 'satisfactory requirements vector' is of considerable interest, particularly in the context analogous to 'Goal Programming' situation, where, in addition to achieving a primary objective, secondary objectives are also considered for achievement.

Thus, for instance, in planning for transport of a material through conduits, basic interest may be in getting the material through a network whose arc capacities may first be fixed on techno-economic considerations pertaining to the overall network system [even otherwise, already a network with given arc capacities may exist] and it may be required to see how far the same network may be made to serve a secondary function of transporting the materials also to the different intermediate nodes, so that new consumption units of appropriate capacities could be installed without detriment to the receipt at the sink.

The situation is somewhat similar to the 'Parametric linear programming', though slightly harder. This problem may be stated as follows:

Find real numbers $\{f_{ij}\}$, to maximise $T_2 = V + \sum_{j=1}^{n} r_j$, where $V = \sum_{j=1}^{n} f_{jn}$, such that

$$0 \leqslant f_{ij} \leqslant C_{ij} \tag{8}$$

$$0 \leqslant r_j \leqslant R_j; \tag{9}$$

and

$$\sum_{i=1}^{n} f_{ij} = r_j + \sum_{k=1}^{n} f_{jk} \tag{10}$$

where $f_{jk} = 0$, $k = 1$ to n; if $r_j \leqslant R_j$ for $j = 2$ to $(n-1)$.

This problem is being investigated in depth and will be reported later. However, to get a feel of the situation, we give below some networks for which different requirements and their effect on the flow to the destination are worked out. The table 11 gives the results of different networks solved on an IRIS-55 computer in FORTRAN-IV language.

Twelve different capacity matrices with different network structures are considered. For each of these capacity matrices two alternatives are considered; (i) with zero requirement vector and (ii) with a specified non-zero requirement vector. Under the second category, three further alternatives are considered as in cases (i), (ii) and (iii) described in the next paragraph.

CASE (i)

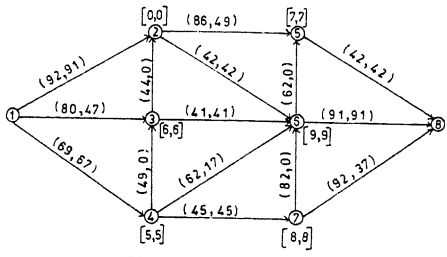
The given MFPR is solved as per the algorithm given in §§ 5 and 6.

CASE (ii)

The MFPR is modified by introducing a pseudo-node $(n+1)$ called the 'super sink', which is connected to (i) node n by a directed arc $(n, n+1)$ with capacity ∞, (ii) the intermediate node j by directed arc $(j, n+1)$ with capacity R_j for $j = 2$ to $(n-1)$.

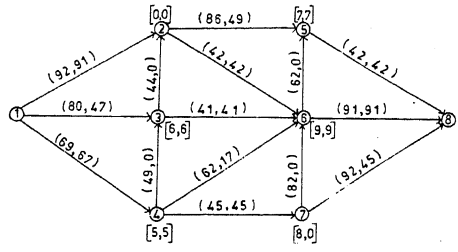
CASE (iii)

The MFPR is modified by introducing two pseudo nodes $(n+1)$ and $(n+2)$ which are connected in the following way: (i) node $(n+2)$ is connected to nodes n and $(n+1)$ by directed arcs $(n, n+2)$ and $(n+1, n+2)$ with capacities equal to ∞; (ii) node $(n+1)$



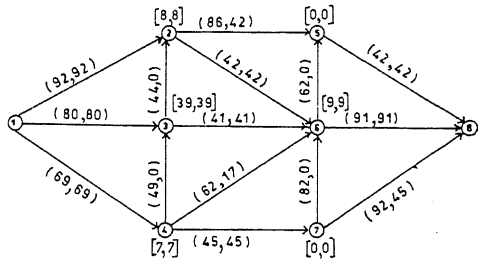
a =205, b =170, c =35, d =35.

Figure 6.



a = 205, b = 178, c = 27 d = 35.

Figure 7.



a = 241, b = 178, c = 63, d = 63.

Figure 8.

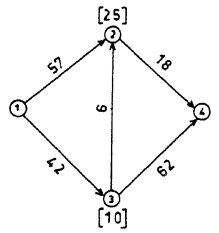


Figure 9.

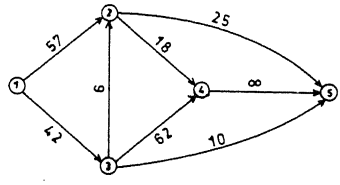


Figure 10.

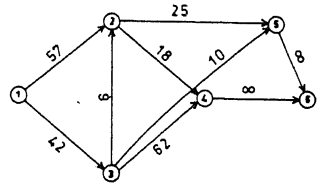


Figure 11.

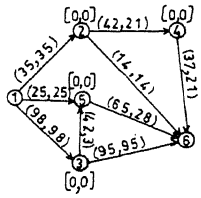
directly connected to an intermediate node j through a directed arc $(j, n+1)$ with capacity R_j for all $j = 2$ to $(n-1)$.

By these modifications cases (ii) and (iii) are converted to equivalent MFP which can be solved by the conventional Ford and Fulkerson algorithm and the solutions can easily be reinterpreted in the context of the cases (ii) and (iii) respectively.

From table 11, it is interesting to note that in most of the cases, we are getting more flow into the sink in case (iii) than in cases (i) and (ii) and the requirements are satisfied less in total amount in case (iii) for the same capacity matrix and the requirements vector than the other two cases.

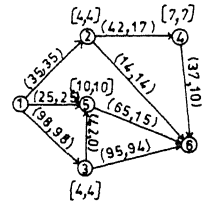
In case (ii) more of a total requirement is satisfied than in case (i) and in no single case the requirement satisfied in case (ii) is less than that of case (i) for the same capacity matrix and the requirements vector, which is not an unexpected result.

The MFPR in case (i) shown in figure 9 will look like as shown in figures 10 and 11 for cases (ii) and (iii) respectively after the modifications. However, in the figures that follow (figures 12–55) later these modifications are not shown because of complications and overlappings of the lines. But the problems are solved as per the modifications only and



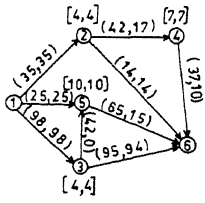
a =158, b =158, c = 0, d = 0.

Figure 12.



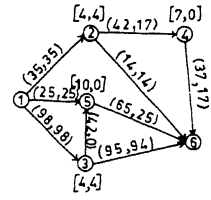
a =158, b =133, c =25, d =25.

Figure 13.



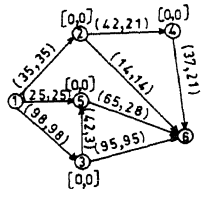
a =158, b =133, c =25, d = 25.

Figure 14.



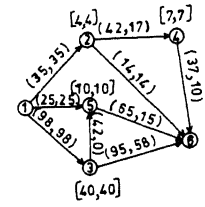
a =158, b =150, c =8, d = 25.

Figure 15.



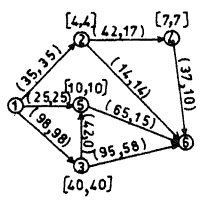
a =158, b =158, c = 0, d = 0.

Figure 16.



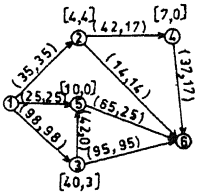
a =158, b =97, c =61, d =61.

Figure 17.



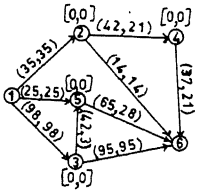
a = 158, b = 97, c = 61, d = 61.

Figure 18.



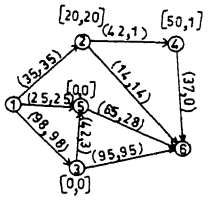
a = 158, b = 151, c = 7, d = 61.

Figure 19.



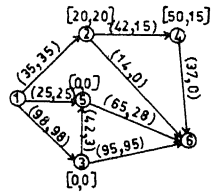
a = 158, b = 158, c = 0, d = 0.

Figure 20.



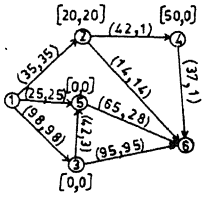
a = 158, b = 137, c = 21, d = 70.

Figure 21.



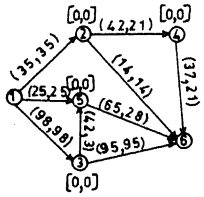
a = 158, b = 123, c = 35, d = 70.

Figure 22.



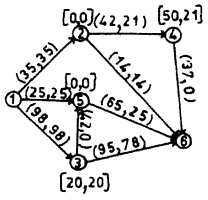
a = 158, b = 138, c = 20, d = 70.

Figure 23.



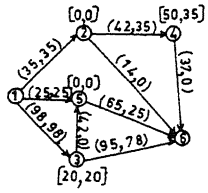
a = 158, b = 158, c = 0, d = 0.

Figure 24.



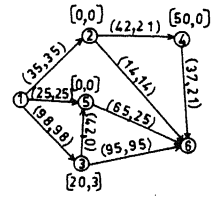
a = 158, b = 117, c = 41, d = 70.

Figure 25.



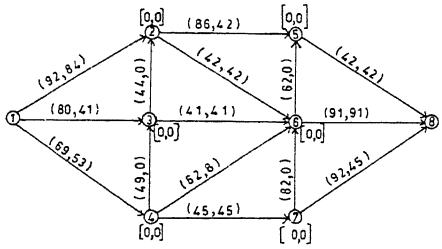
a = 158, b = 103, c = 55, d = 70.

Figure 26.



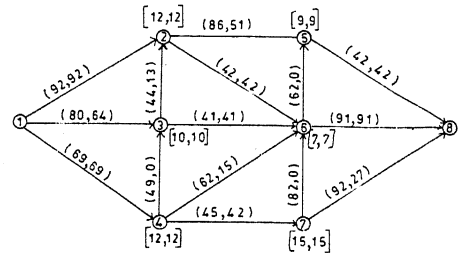
a = 158, b = 155, c = 3, d = 70

Figure 27.



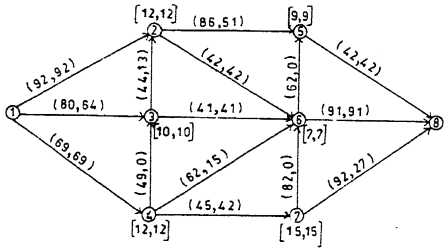
a = 178, b = 178, c = 0, d = 0.

Figure 28.



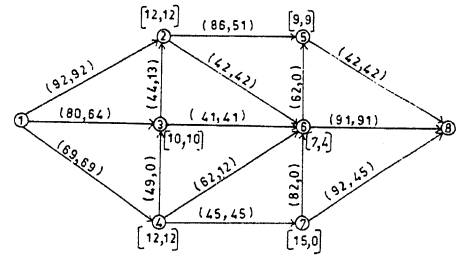
a = 225, b = 160, c = 65, d = 65.

Figure 29.



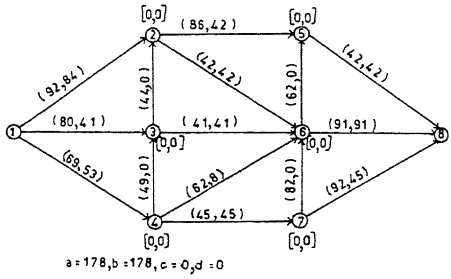
a = 225, b = 160, c = 65, d = 65.

Figure 30.



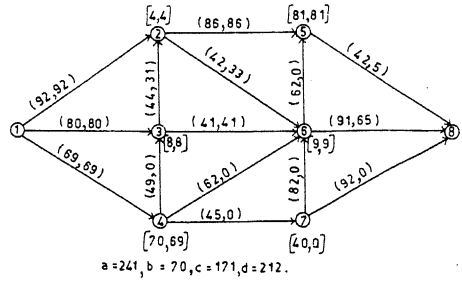
a = 225, b = 178, c = 47, d = 65.

Figure 31.



a = 178, b = 178, c = 0, d = 0

Figure 32.



a = 241, b = 70, c = 171, d = 212.

Figure 33.

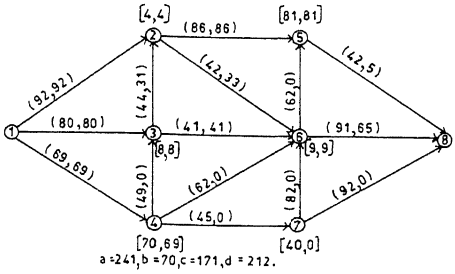


Figure 34.

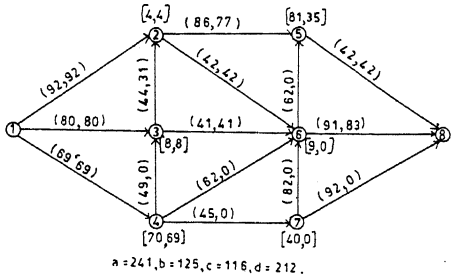
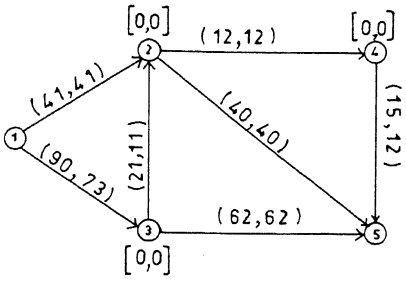
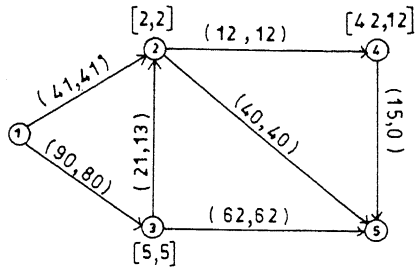


Figure 35.



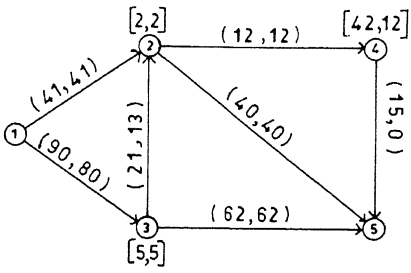
a=114, b=114, c=0, d=0.

Figure 36.



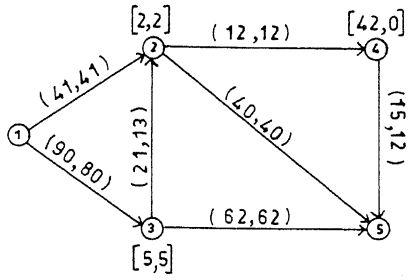
a=121, b=102, c=19, d=49.

Figure 37.



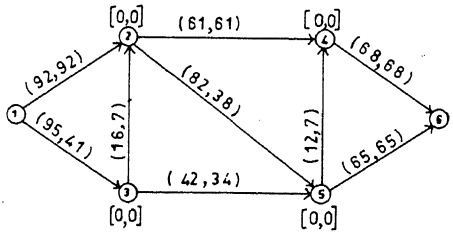
a=121, b=102, c=19, d=49.

Figure 38.



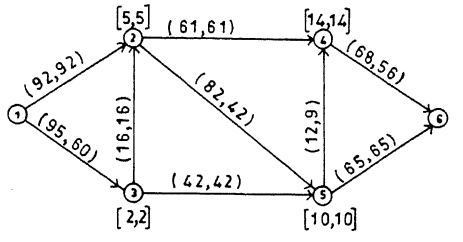
a=121, b=114, c=7, d=49.

Figure 39.



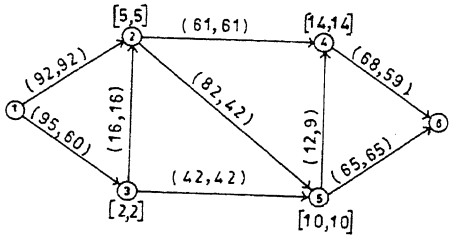
a=133, b=133, c=0, d=0.

Figure 40.



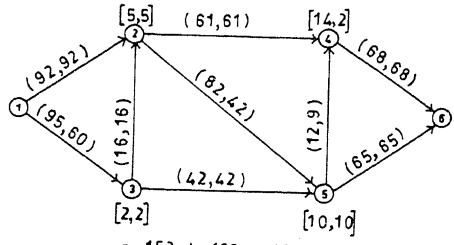
a=152, b=121, c=31, d=31.

Figure 41.



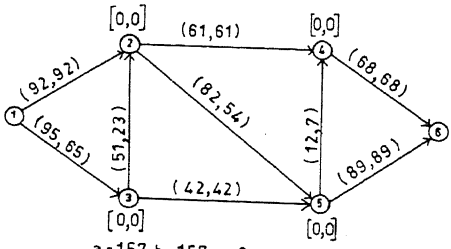
a = 152, b = 121, c = 31, d = 31

Figure 42.



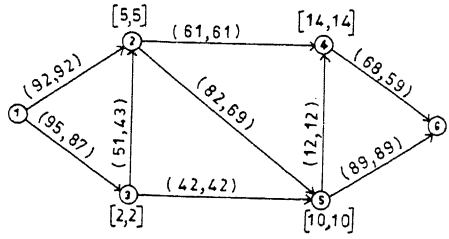
a = 152, b = 133, c = 19, d = 31

Figure 43.



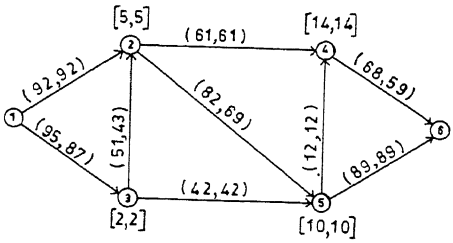
a = 157, b = 157, c = 0, d = 0.

Figure 44.



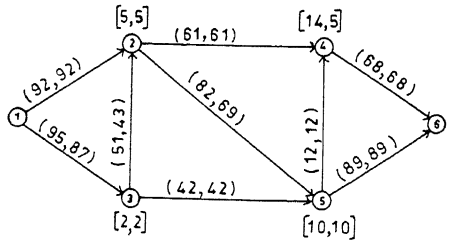
a = 179, b = 148, c = 31, d = 31.

Figure 45.



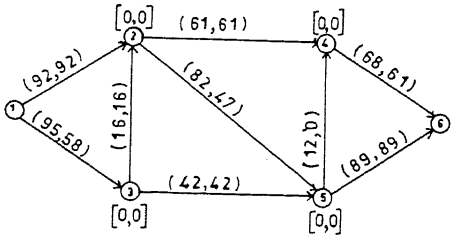
a = 179, b = 148, c = 31, d = 31.

Figure 46.



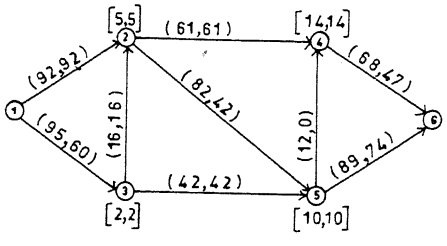
a = 179, b = 157, c = 22, d = 31.

Figure 47.



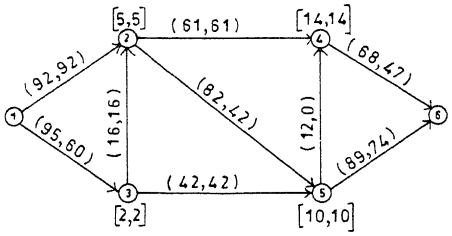
a = 150, b = 150, c = 0, d = 0.

Figure 48.



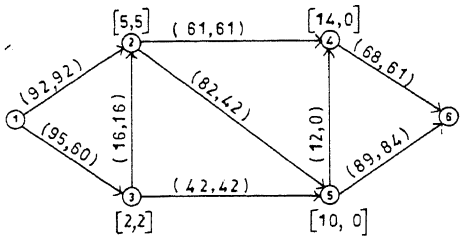
a = 152, b = 121, c = 31, d = 31

Figure 49.



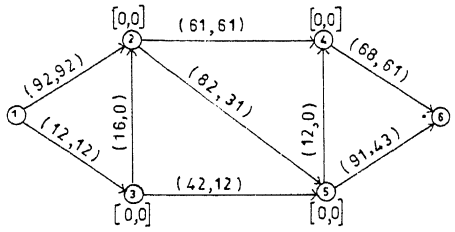
a =152, b =121, c =31, d =31.

Figure 50.



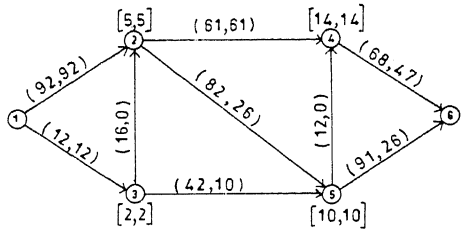
a =152, b =145, c = 7, d =31.

Figure 51.



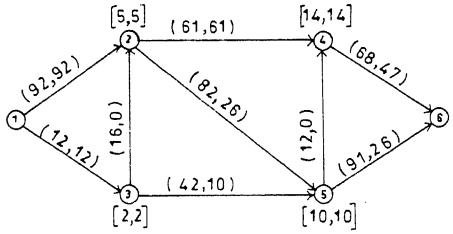
a = 104, b = 104, c = 0, d = 0.

Figure 52.



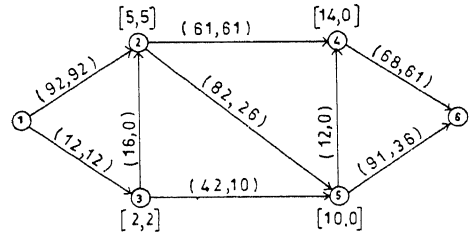
a =104, b = 73, c =31, d = 31.

Figure 53.



a=104, b=73, c=31, d=31.

Figure 54.



a = 104, b =97, c = 7, d = 31.

Figure 55.

the flow values (the requirements met or satisfied) shown besides the nodes in square brackets such that the first number is the requirement and the second number is the flow (requirement met).

Table 12. A check list of figures for different cases.

	Capacity matrix											
Case	1	2	3	4	5	6	7	8	9	10	11	12
MFP	4	12	16	20	24	28	32	36	40	44	48	52
Case (i)	5	13	17	21	25	29	33	37	41	45	49	53
Case (ii)	6	14	18	22	26	30	34	38	42	46	50	54
Case (iii)	7	15	19	23	27	31	35	39	43	47	51	55

For each of the four cases, with each of the twelve capacity matrices, the corresponding network diagram with the optimal flows and other relevant details are given in the different figures as listed in table 12.

Acknowledgement

BKR thanks the CSIR for financial assistance in the form of JRF.

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On Erdös-Lax theorem

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MS received 19 December 1983

Abstract. An improved version of the generalized Erdös-Lax theorem is stated and proved.

Keywords. Erdös-Lax theorem; generalization; maximum modulus principle; zero of polynomials.

1. Introduction

Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n and $P'(z)$ denote its derivative. Let also $\|P\| = \max_{z=1} |P(z)|$ and $\|P'\| = \max_{z=1} |P'(z)|$. Concerning the estimate of $\|P'\|$, the following result is known.

THEOREM A

When all the zeros of $P(z)$ lie on $|z| = K > 1$, then

$$\frac{n}{1+K^n}\|P\| \leqslant \|P'\| \leqslant \frac{n}{1+K}\|P\|. \tag{1}$$

The upper bound for $k = 1$ is now known as Erdös-Lax Theorem (Lax [4]). For the general case, see Govil [2], Govil and Rahman [3], Malik [5] and Turan [6]. We obtain an improvement of this theorem.

2. Proof

THEOREM 1

If $P(z)$ has no zeros in $|z| < K$, $K \geqslant 1$ and $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$ then

$$\|P'\| \leqslant \frac{n}{1+K^\mu}\|P\|. \tag{2}$$

There is equality in (2) for $P(z) = (z^\mu + K^\mu)^{n/\mu}$ where n is a multiple of μ.

Proof of Theorem 1. Since $P(z) \neq 0$ in $|z| < K$ from Laguerre's Theorem [1] we

have

$$\xi P'(z) \neq zP'(z) - nP(z) \tag{3}$$

for $|\xi| < K, |z| < K$. For any fixed z, we can appropriately choose the arg ξ in (3) to get

$$|\xi|\,|P'(z)| \neq |zP'(z) - nP(z)| \tag{4}$$

From where we have

$$|\xi|\,|P'(z)| < |zP'(z) - nP(z)| \tag{5}$$

for $|\xi| < K, |z| < K$ because the otherwise inequality is violated for sufficiently small values of $|\xi|$. Making $|\xi| \to K$ and $|z| \to K$ in (5) one has

$$K|P'(Kz)| \leqslant |KzP'(Kz) - nP(Kz)| \tag{6}$$

for $|z| \leqslant 1$. Since $c_1 = c_2 = \ldots = c_{\mu-1} = 0$, from. (6) we get

$$K^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_{\nu}(Kz)^{\nu-\mu}\right| \leqslant |KzP'(Kz) - nP(Kz)| \tag{7}$$

for $|z| \leqslant 1$. In fact (7) holds for $|z| = 1$ and for $\rho < K$ instead of K. But $\rho zP'(\rho z) - nP(\rho z) \neq 0$ in $|z| < 1$, by the maximum modulus principle it also holds for $|z| < 1$ and then let $\rho \to K$. Taking z/K instead of z in (7) we have

$$K^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_{\nu} z^{\nu-1}\right| \leqslant |zP'(z) - nP(z)| \tag{8}$$

for $|z| = 1$. Consequently

$$K^{\mu}|P'(z)| \leqslant |zP'(z) - nP(z)| \tag{9}$$

for $|z| = 1$. On the other hand, from the result of De Bruijn [1]: for $|\xi| \leqslant 1, |z| \leqslant 1$,

$$\xi\frac{P'(z)}{n} - \frac{zP'(z)}{n} + P(z) \in S \tag{10}$$

where $S = \{P(z): |z| \leqslant 1\}$; hence for $|z| = 1$

$$|P'(z)| + |zP'(z) - nP(z)| \leqslant n\|P\|. \tag{11}$$

From (9) and (11), one has

$$(1 + K^{\mu})\|P'\| \leqslant n\|P\|.$$

Consequently (2).

If $P(z) = \sum_{\nu=0}^{n-\mu} c_{\nu} z^{\nu} + c_n z^n$ has all its zeros in the disk $|z| \leqslant k \leqslant 1$ then $Q(z) = z^n P(1/z)$ satisfies the hypothesis of theorem 1 with $K = 1/k$. Since

$$Q'(z) = nz^{n-1}P(1/z) - z^{n-2}P'(1/z). \tag{12}$$

We conclude that

$$\begin{aligned}\|P'\| &\geqslant n\|P\| - \|Q'\| \\ &\geqslant n\|P\| - \frac{nk^{\mu}}{1+k^{\mu}}\|P\| = \frac{n}{1+k^{\mu}}\|P\|.\end{aligned} \tag{13}$$

Thus we have the following:

COROLLARY

If $P(z) = \sum_{\nu=0}^{n-\mu} c_\nu z^\nu + c_n z^n$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$ then

$$\|P'\| \geqslant \frac{n}{1+k^\mu} \|P\|. \tag{14}$$

There is equality in (14) for $P(z) = (z^\mu + k^\mu)^{n/\mu}$ where n is a multiple of μ.

3. Conclusion

Finally, as an application of Erdös-Lax Theorem we present the following:

THEOREM 2

If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ with $c_n = 1$ has no zeros in $|z| < 1$ and $\|P\| = 2$, then $P(z) = z^n + \alpha$ where $|\alpha = 1$.

Proof of Theorem 2. Since $\|P\| = 2$ and $P(z)$ has no zeros in $|z| < 1$, from theorem A we have $|P'(z)| \leqslant n$ for $|z| \leqslant 1$. Using a result due to Visser [7]: if $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$, then $|c_0| + |c_n| \leqslant \|P\|$, we get $n + |c_1| \leqslant n$. But this implies that $c_1 = 0$ and so $P'(z) = \sum_{\nu=2}^{n} \nu c_\nu z^{\nu-1}$. Applying that result again to $\frac{P'(z)}{z} = \sum_{\nu=2}^{n} \nu c_\nu z^{\nu-2}$, one concludes that $c_2 = 0$. Consequently, all the coefficients $c_\nu = 0$ except c_n and c_0. Since $c_n = 1$, one must have $c_0 = \alpha$, where $|\alpha| = 1$ because $\|P\| = 2$.

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